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CHAPTER 1

Module 1: Ratios and Unit Rates

Chapter Outline

1.1 Forms of Ratios
1.2 Equivalent Ratios
1.3 Ratios in Simplest Form
1.4 Comparison of Ratios in Decimal Form
1.5 Proportion Properties
1.6 Identification and Writing of Equivalent Rates
1.7 Comparison of Unit Rates
1.8 Unit Rates
1.9 Indirect Measurement
1.10 Function Rules for Input-Output Tables
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1.12 Rates of Change
1.13 Percent Equations
1.14 Use the Percent Equation to Find Part a
1.15 Use the Percent Equation to Find the Percent
1.16 Use the Percent Equation to Find the Base, b
1.17 Dimensional Analysis

Understand ratio concepts and use ratio reasoning to solve problems.

6.RP.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, “The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak.” “For every vote candidate A received, candidate C received nearly three votes.”

6.RP.2 Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. For example, “This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar.” “We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger.”

6.RP.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

a. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

b. Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.
1.1 Forms of Ratios

Here you’ll learn what a ratio is and the different forms it can take. You’ll also learn how to find and reduce ratios and how to convert measurements using ratios.

**Ratios**

A *ratio* is a way to compare two numbers. Ratios can be written in three ways: \( \frac{a}{b} \), \( a : b \), and \( a \) to \( b \).

We always reduce ratios just like fractions. When two or more ratios reduce to the same ratio they are called *equivalent ratios*. For example, 50:250 and 2:10 are *equivalent ratios* because they both reduce to 1:5.

One common use of ratios is as a way to convert measurements.

What if you were told that there were 15 birds on a pond and that the ratio of ducks to geese is 2:3? How could you determine how many ducks and how many geese are on the pond?

**Examples**

For Examples 1 and 2, use the following information:

The total bagel sales at a bagel shop for Monday is in the table below.

<table>
<thead>
<tr>
<th>Type of Bagel</th>
<th>Number Sold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plain</td>
<td>80</td>
</tr>
<tr>
<td>Cinnamon Raisin</td>
<td>30</td>
</tr>
<tr>
<td>Sesame</td>
<td>25</td>
</tr>
<tr>
<td>Jalapeno Cheddar</td>
<td>20</td>
</tr>
<tr>
<td>Everything</td>
<td>45</td>
</tr>
<tr>
<td>Honey Wheat</td>
<td>50</td>
</tr>
</tbody>
</table>

**Example 1**

What is the ratio of honey wheat bagels to total bagels sold?

Order matters. Honey wheat is listed first, so that number comes first in the ratio (or on the top of the fraction). Find
the total number of bagels sold, \(80 + 30 + 25 + 20 + 45 + 50 = 250\).
The ratio is \(\frac{50}{250} = \frac{1}{5}\).

**Example 2**

What is the ratio of cinnamon raisin bagels to sesame bagels to jalapeno cheddar bagels?
You can have ratios that compare more than two numbers and they work just the same way. The ratio for this problem is 30:25:20, which reduces to 6:5:4.

**Example 3**

There are 14 girls and 18 boys in your math class. What is the ratio of girls to boys?
Remember that order matters. The question asked for the ratio of girls to boys. The ratio would be 14:18. This can be simplified to 7:9.

**Example 4**

Simplify the following ratios.

a. \(\frac{7\ ft}{14\ in}\)

Change the ratio so that each part is in the same units. Remember that there are 12 inches in a foot.

\[
\frac{7\ ft}{14\ in} \cdot \frac{12\ in}{1\ ft} = \frac{84}{14} = \frac{6}{1}
\]

The inches and feet cancel each other out. Simplified ratios do not have units.

b. \(9m:900cm\)

Change the ratio so that each part is in the same units. It is easier to simplify a ratio when written as a fraction.

\[
\frac{9\ m}{900\ cm} \cdot \frac{100\ cm}{1\ m} = \frac{900}{900} = \frac{1}{1}
\]

c. \(\frac{4\ gal}{16\ gal}\)

Change the ratio so that each part is in the same units.

\[
\frac{4\ gal}{16\ gal} = \frac{1}{4}
\]
1.1. Forms of Ratios

Example 5

A talent show has dancers and singers. The ratio of dancers to singers is 3:2. There are 30 performers total, how many of each are there?

To solve, notice that 3:2 is a reduced ratio, so there is a number, \( n \), that we can multiply both by to find the total number in each group. Represent dancers and singers as expressions in terms of \( n \). Then set up and solve an equation.

\[
dancers = 3n, \quad singers = 2n \quad \rightarrow \quad 3n + 2n = 30
\]

\[
5n = 30
\]

\[
n = 6
\]

There are \( 3 \cdot 6 = 18 \) dancers and \( 2 \cdot 6 = 12 \) singers.

Review

1. The votes for president in a club election were: Smith : 24 Munoz : 32 Park : 20Find the following ratios and write in simplest form.
   - Votes for Munoz to Smith
   - Votes for Park to Munoz
   - Votes for Smith to total votes
   - Votes for Smith to Munoz to Park

Use the picture to write the following ratios for questions 2-6.

\[
AEFD \text{ is a square} \quad ABCD \text{ is a rectangle}
\]

2. \( AE : EF \)
3. \( EB : AB \)
4. \( DF : FC \)
5. \( EF : BC \)
6. Perimeter \( ABCD \) : Perimeter \( AEFD \) : Perimeter \( EBCF \)

Simplify the following ratios. Remember that there are 12 inches in a foot, 3 feet in a yard, and 100 centimeters in a meter.

7. \( \frac{25 \text{ in}}{5 \text{ ft}} \)
8. \( \frac{9 \text{ ft}}{3 \text{ yd}} \)
9. \( \frac{95 \text{ cm}}{1.5 \text{ m}} \)

10. The measures of the angles of a triangle are in the ratio 3:3:4. What are the measures of the angles?
11. The length and width of a rectangle are in a 3:5 ratio. The perimeter of the rectangle is 64. What are the length and width?
12. The length and width of a rectangle are in a 4:7 ratio. The perimeter of the rectangle is 352. What are the length and width?
13. A math class has 36 students. The ratio of boys to girls is 4:5. How many girls are in the class?
14. The senior class has 450 students in it. The ratio of boys to girls is 8:7. How many boys are in the senior class?
15. The varsity football team has 50 players. The ratio of seniors to juniors is 3:2. How many seniors are on the team?

**Review (Answers)**

To see the Review answers, open this [PDF file](#) and look for section 7.1.

**Resources**

Click image to the left or use the URL below.

URL: [https://www.ck12.org/flx/render/embeddedobject/1358](https://www.ck12.org/flx/render/embeddedobject/1358)
Here you’ll learn to identify and write equivalent ratios.

Have you ever been sent on an errand? Casey is off to the grocery store. Take a look.

On the way home from soccer practice, Casey’s mom sends her into the grocery store to get a half gallon of milk. Casey is hungry after practice, so she isn’t paying attention to what kind her Mom has asked her to get. In Casey’s house they drink both whole milk and skim milk. Casey runs to the dairy section of the grocery store and stops short. She isn’t sure what to get. There are five different kinds of milk. There is whole milk, reduced fat milk, lowfat milk, skim milk and organic milk. There are also different brands to choose from: Hood, Eagle Brand, and Garelic for non-organic milk, and Organic Valley and Nature’s Valley for organic brands. Casey notices that there are three non-organic brands to the two organic brands. Then she notices that the supermarket has its own brand of non-organic milk as well. That makes four non-organic brands to two organic brands. Casey is sure that this means something. She has been reading about organic food in school and is interested in organic milk and food. Casey wishes that there were more organic brands than non organic brands. She decides to make a note of this to show her teacher at school.

If Casey wants to document this information as a ratio, how could she do it?

This Concept will teach you all you need to know about equivalent ratios.

**Guidance**

This Concept focuses on **ratios**. Ratios are everywhere in everyday life. In fact, we work with ratios so much that we probably don’t even realize that we are working with them. In this Concept, you will learn how to write ratios, simplify ratios and compare ratios, but there is a question that we must answer first.

**What is a ratio?**

A **ratio** is a comparison of two quantities. The quantities can be nearly anything: people, cars, dollars... even two groups of things!

Let’s look at a picture.
We can write ratios that compare the boys in this picture, but how?

**How do we write a ratio?**

A ratio is written in three different ways. It can be written as a fraction, with the word “to” or with a colon.

Let’s take a look at this in action by writing ratios that compare the boys in the picture.

What is the ratio of boys with striped shirts to boys with solid shirts? There are two boys with striped shirts and two boys with solid shirts. **Let’s write the ratio in three ways.**

\[
\begin{align*}
\frac{2}{2} \\
2 : 2 \\
2 \text{ to } 2
\end{align*}
\]

Each of these ratios is correct. Notice that we are comparing an individual quality here.

**What about comparing a category to the group?**

What is the ratio of boys that are holding binders to all of the boys?

There are two boys holding binders and four boys in the group. **Let’s write the ratio three different ways.** Notice that the first thing being compared comes first when writing the ratio. Or the first thing becomes the numerator in the fraction form of the ratio.

\[
\begin{align*}
\frac{2}{4} \\
2 : 4 \\
2 \text{ to } 4
\end{align*}
\]

Each of these ratios is *equivalent*, meaning that they are all equal. Each ratio, though written in a different form, is an equivalent ratio.

We can write many different ratios by comparing these figures. Let’s list some and use the word “to” to write our ratio form.

Stars to circles = 3 to 2
Red stars to total stars = 2 to 3
Red stars to blue stars = 2 to 1
Blue stars to red stars = 1 to 2
Blue stars to total stars = 1 to 3

We could continue making this list.

We can also write the same ratios using a colon or a fraction.

Practice on your own. Use the picture to write each ratio three different ways.

Example A
What is the ratio of orange marbles to green marbles?
Solution: 2 to 4

Example B
What is the ratio of yellow marbles to total marbles?
Solution: 2 to 22

Example C
What is the ratio of orange marbles to total marbles?
Solution: 2 to 22

Now we can help with the milk comparisons. These comparisons can be written as ratios.

If Casey wants to document this information as a ratio, how could she do it?

A ratio is a comparison. We can write a ratio to compare two quantities in three different ways. In this problem, Casey wants to compare organic and non-organic brands of milk.

She notices that there are four non-organic brands and two organic brands.
Casey can write this comparison three different ways.

\[
\begin{align*}
4 & \text{ to } 2 \\
4 & \div 2 \\
4 : 2
\end{align*}
\]

This is our answer.

**Guided Practice**

Here is one for you to try on your own.

What is the ratio of total marbles to dark blue marbles?

**Answer**

First, count the total number of marbles. There are 22 marbles.

There are three dark blue marbles.

Our answer is 22 to 3.

**Video Review**

Khan Academy, Introduction to Ratios

[Link](https://www.ck12.org/flx/render/embeddedobject/5407)

James Sousa, Introduction to Ratios

[Link](https://www.ck12.org/flx/render/embeddedobject/1358)
1.2. Equivalent Ratios

Explore More

Directions: Use the picture to answer the following questions. Write each ratio three ways.

1. What is the ratio of hens to chicks?
2. What is the ratio of green chicks to yellow chicks?
3. What is the ratio of white chicks to total chicks?
4. What is the ratio of green chicks to total chicks?
5. What is the ratio of yellow chicks to total chicks?
6. What is the ratio of green chicks to white chicks?

7. What is the ratio of light blue marbles to dark blue ones?
8. What is the ratio of orange marbles to red marbles?
9. What is the ratio of pink marbles to red marbles?
10. What is the ratio of green marbles to total marbles?
11. What is the ratio of yellow marbles to red marbles?
12. What is the ratio of total marbles to dark purple marbles?
13. What is the ratio of total marbles to all blue marbles?
14. What is the ratio of pink marbles to total marbles?
15. What is the ratio of red marbles to total marbles?
**1.3 Ratios in Simplest Form**

In this concept, you will learn how to simplify ratios and then compare and draw conclusions.

Grace is working on an assignment. She has to survey 50 people and ask if they are either right-handed or left-handed. Out of the 50 people she surveyed, 5 were left-handed and 45 were right-handed. How can Grace use this information to describe the results of her survey?

In this concept, you will learn how to simplify ratios and then compare and draw conclusions.

**Simplifying Ratios**

A ratio is the comparison of two quantities. Ratios can involve large quantities that may not represent a clear comparison. Simplify ratios to make them easier to evaluate. Since ratios can also be written as fractions, you can simplify ratios the same way you simplify fractions.

Let’s look at the ratio of left-handed people to right-handed people. The ratio of left-handed people to right-handed people is 5 to 45. First, write the ratio using the fraction notation.

\[
\frac{5}{45}
\]

Then, find the greatest common factor (GCF) to find the simplest form of the ratio. The GCF is the largest factor share by both numbers. The GCF of 5 and 45 is 5.

Next, divide the numerator and the denominator by the GCF.

\[
\frac{5 \div 5}{45 \div 5} = \frac{1}{9}
\]
The simplest form of the ratio \( \frac{5}{45} \) is \( \frac{1}{9} \), which can also be written as 1 to 9 or 1 : 9.

There is one left-handed person for every nine right-handed people.

Remember, when you simplify a ratio, the value of the ratio does not change. Therefore, a ratio and its simplest form are equivalent ratios.

\[
\frac{5}{45} = \frac{1}{9}
\]

**Examples**

**Example 1**

Earlier, you were given a problem about Grace’s survey.

Of the 50 people she surveyed, 5 were left-handed and 45 were right-handed. Use a different ratio. Simplify the ratio and draw a conclusion.

First, decide which ratio to use and write it as a fraction. Let’s use the ratio of left-handed people to the total number of people surveyed.

\[
\frac{5}{50}
\]

Next, find the GCF. The GCF of 5 and 50 is 5.

Then, divide the numerator and the denominator by 5.

\[
\frac{5}{50} = \frac{1}{10}
\]

The simplest form of \( \frac{5}{50} \) is \( \frac{1}{10} \).

There is one left-handed person for every ten people surveyed.

**Example 2**

Simplify the following ratio: \( \frac{12}{18} \). Write the simplified ratio as a fraction.

First, find the GCF of 12 and 18. The GCF is 6.

Next, divide the numerator and the denominator by 6.

\[
\frac{12}{18} = \frac{2}{3}
\]

The simplest form of \( \frac{12}{18} \) is \( \frac{2}{3} \).

**Example 3**

Simplify the following ratio: \( \frac{2}{16} \). Write the simplified ratio as a fraction.
First, find the GCF of 2 and 10. The GCF is 2.
Next, divide the numerator and denominator by 2.

\[
\frac{2 \div 2}{10 \div 2} = \frac{1}{5}
\]

The simplest form of \(\frac{2}{10}\) is \(\frac{1}{5}\).

**Example 4**

Simplify the following ratio: 6 to 8. Write the simplified ratio as a fraction.
First, find the GCF of 6 and 8. The GCF is 2.
Next, divide the both numbers by 2.

\[
\frac{6 \div 2}{8 \div 2} = \frac{3}{4}
\]

Then, write 3 to 4 as a fraction.

\[
\frac{3}{4}
\]

The simplest form of 6 to 8 is 3 to 4 or \(\frac{3}{4}\).

**Example 5**

Simplify the following ratio: 5 : 20. Write the simplified ratio as a fraction.
First, find the GCF for 5 and 20. The GCF is 5.
Next, divide both numbers by 5.

\[
\frac{5 \div 5}{20 \div 5} = \frac{1}{4}
\]

Then, write 1 : 4 as a fraction.

\[
\frac{1}{4}
\]

The simplest form of 5 : 20 is 1 : 4 or \(\frac{1}{4}\).
Review

Find the simplest form for each ratio. Write your answer as a fraction.

1. 2 to 4
2. 3 : 6
3. 5 to 15
4. 2 to 30
5. 10 to 15
6. \( \frac{4}{6} \)
7. 3 : 9
8. 6 : 8
9. \( \frac{2}{8} \)
10. \( \frac{4}{15} \)
11. 10 to 12
12. 7 : 21
13. 12 : 24
14. 25 to 75
15. \( \frac{27}{30} \)
16. \( \frac{48}{80} \)
17. \( \frac{18}{80} \)

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.2.

Resources

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/181827
Here you’ll learn to write and compare ratios in decimal form.

Remember Casey and the milk comparison in the Ratios in Simplest Form Concept? Well, look at what she is up to now.

Casey decided to hold a survey about the milk choices of customers at the supermarket. She discovers that many people purchase regular milk and half as many purchase organic milk. Casey surveyed 50 people.

Here is what she found.

35 out of 50 purchased regular milk.
15 out of 50 purchased organic milk.

If Casey wanted to think about these ratios as decimals, could she do it? What would the decimals be for each choice?

This Concept will teach you how to do these conversions.

Guidance

Previously we worked on writing ratios in fraction form and simplifying them. What about decimal form? Fractions and decimals are related, in fact a fraction can be written as a decimal and a decimal can be written as a fraction.

Is it possible to write a ratio as a decimal too?

Yes! Because a ratio can be written as a fraction, it can also be written as a decimal. To do this, you will need to remember how to convert fractions to decimals.

Now we can apply this information to our work with ratios.

Convert 2:4 into a decimal.

First, write it as a ratio in fraction form.

\[
2 : 4 = \frac{2}{4}
\]
Next, simplify the fraction if possible.

\[
\frac{2}{4} = \frac{1}{2}
\]

Finally, convert the fraction to a decimal.

\[
\frac{5}{2)1.0}
\]

Our answer is .5.

Practice by converting each ratio to decimal form.

**Example A**

4 to 5
Solution: .80

**Example B**

\[
\frac{5}{20}
\]
Solution: .25

**Example C**

6 to 10
Solution: .60

Now let’s go and help Casey convert her ratios into decimal form. Here is the original problem once again.

Remember Casey and the milk comparison? Well, look at what she is up to now.

Casey decided to hold a survey about the milk choices of customers at the supermarket. She discovers that many people purchase regular milk and half as many purchase organic milk. Casey surveyed 50 people.

Here is what she found.

35 out of 50 purchased regular milk.
15 out of 50 purchased organic milk.
If Casey wanted to think about these ratios as decimals, could she do it? What would the decimals be for each choice?

First, we can write a ratio in fraction form. We can use convert the ratios.

\[
\frac{35}{50} = \frac{70}{100} \\
\frac{15}{50} = \frac{30}{100}
\]

The first decimal is .70.

The second decimal is .30.

**Guided Practice**

Here is one for you to try on your own.

Write 2 out of 25 as a decimal.

**Answer**

To write this ratio as a decimal, we can use a denominator of 100 and create equal fractions.

\[
\frac{2}{25} = \frac{?}{100}
\]

Next, we figure out the unknown quantity.

25 times 4 = 100

2 times 4 = 8

\[
\frac{8}{100}
\]

Our answer is .08.

**Video Review**

[James Sousa, Introduction to Ratios](https://www.ck12.org/flx/render/embeddedobject/1358)

[James Sousa, Example of Writing a Ratio as a Simplified Fraction](https://www.ck12.org/flx/render/embeddedobject/5408)
James Sousa, Another Example of Writing a Ratio as a Simplified Fraction

**Explore More**

**Directions:** Convert the following ratios into decimals.

1. 3 to 4
2. 2 to 4
3. \(\frac{1}{3}\)
4. 25 to 100
5. 16 to 32
6. 4 out of 5
7. 6 out of 20
8. \(\frac{1}{4}\)
9. 5 to 6
10. 1:2
11. 4:10
12. 10:50
13. 75 to 100
14. 1 to 3
15. 6 to 8
1.5 Proportion Properties

Here you’ll learn what a proportion is, the properties of proportions, and how to solve proportions using cross-multiplication.

**Proportions**

A proportion is two ratios that are set equal to each other. Usually the ratios in proportions are written in fraction form. An example of a proportion is $\frac{2}{3} = \frac{5}{10}$. To solve a proportion, you need to cross-multiply. The Cross-Multiplication Theorem, which allows us to solve proportions using this method, states that if $a, b, c,$ and $d$ are real numbers, with $b \neq 0$ and $d \neq 0$ and if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. Cross-multiplying allows us to get rid of the fractions in our equation. The Cross-Multiplication Theorem has several sub-theorems, called corollaries.

**Corollary #1:** If $a, b, c,$ and $d$ are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Switch $b$ and $c$.

**Corollary #2:** If $a, b, c,$ and $d$ are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$.

Switch $a$ and $d$.

**Corollary #3:** If $a, b, c,$ and $d$ are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{c}{d}$.

Flip each ratio upside down.

**Corollary #4:** If $a, b, c,$ and $d$ are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

**Corollary #5:** If $a, b, c,$ and $d$ are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

What if you were told that a scale model of a python is in the ratio of 1:24? If the model measures 0.75 feet long, how long is the real python?

**Examples**

**Example 1**

In the picture, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

Find the measures of $AC$ and $XY$. 

![Image](https://www.ck12.org/flx/render/embeddedobject/136691)
Plug in the lengths of the sides we know.

\[
\frac{4}{XY} = \frac{3}{9} \quad \text{and} \quad \frac{3}{9} = \frac{AC}{15}
\]

36 = 3(\text{XY})

\[\text{XY} = 12\]

3 = \frac{AC}{15}

9(\text{AC}) = 45

\[\text{AC} = 5\]

**Example 2**

In the picture, \(\frac{AB}{BE} = \frac{AC}{CD}\). Find \(BE\).

Substitute in the lengths of the sides we know.

\[
\frac{12}{BE} = \frac{20}{25} \quad \rightarrow \quad 20(\text{BE}) = 12(25)
\]

\[\text{BE} = 15\]

**Example 3**

Solve the proportions. Remember, to solve a proportion, you need to cross-multiply.

a. \(\frac{4}{5} = \frac{x}{30}\)

\[
4 \cdot 30 = 5 \cdot x \\
120 = 5x \\
24 = x
\]
b. \( \frac{y+1}{8} = \frac{5}{20} \)

\[
\frac{y+1}{8} = \frac{5}{20} \\
(y+1) \cdot 20 = 5 \cdot 8 \\
20y + 20 = 40 \\
20y = 20 \\
y = 1
\]

c. \( \frac{6}{5} = \frac{2x+4}{x-2} \)

\[
\frac{6}{5} = \frac{2x+4}{x-2} \\
6 \cdot (x-2) = 5 \cdot (2x+4) \\
6x - 12 = 10x + 20 \\
-32 = 4x \\
-8 = x
\]

**Example 4**

Your parents have an architect’s drawing of their home. On the paper, the house’s dimensions are 36 in by 30 in. If the shorter length of the house is actually 50 feet, what is the longer length?

To solve, first set up a proportion. If the shorter length is 50 feet, then it lines up with 30 in, the shorter length of the paper dimensions.

\[
\frac{30}{36} = \frac{50}{x} \\
30x = 1800 \\
x = 60 \quad \text{The longer length is 60 feet.}
\]

**Example 5**

Suppose we have the proportion \( \frac{2}{5} = \frac{14}{35} \). Write three true proportions that follow.

First of all, we know this is a true proportion because you would multiply \( \frac{2}{5} \) by \( \frac{7}{7} \) to get \( \frac{14}{35} \). Using the first three corollaries:

1. \( \frac{2}{14} = \frac{5}{35} \)
2. \( \frac{35}{14} = \frac{14}{7} \)
3. \( \frac{5}{2} = \frac{35}{14} \)

**Review**

Solve each proportion.
1. \( \frac{x}{10} = \frac{42}{35} \)
2. \( \frac{x}{7} = \frac{3}{7} \)
3. \( \frac{6}{y} = \frac{3}{3} \)
4. \( \frac{3}{y} = \frac{16}{x} \)
5. \( \frac{y-3}{2} = \frac{y+6}{7} \)
6. \( \frac{8}{22} = \frac{16}{7} \)

7. Shawna drove 245 miles and used 8.2 gallons of gas. At the same rate, if she drove 416 miles, how many gallons of gas will she need? Round to the nearest tenth.

8. The president, vice-president, and financial officer of a company divide the profits in a 4:3:2 ratio. If the company made $1,800,000 last year, how much did each person receive?

Given the true proportion, \( \frac{10}{6} = \frac{15}{d} = \frac{x}{y} \) and \( d, x, \) and \( y \) are nonzero, determine if the following proportions are also true.

9. \( \frac{10}{y} = \frac{x}{6} \)
10. \( \frac{15}{10} = \frac{d}{6} \)
11. \( \frac{6+10}{x} = \frac{y+x}{y} \)
12. \( \frac{12}{x} = \frac{y}{d} \)

For questions 13-16, \( \frac{AE}{ED} = \frac{BC}{CD} \) and \( \frac{ED}{AD} = \frac{CD}{DB} = \frac{EC}{AB} \).

13. Find \( DB \).
14. Find \( EC \).
15. Find \( CB \).
16. Find \( AD \).

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 7.2.

**Resources**

**MEDIA**

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/1359
MEDIA

Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/1361
Here you’ll learn to identify and write equivalent rates.

Have you ever bought things in bulk at a supermarket? People do it all the time, and you need to know how to work with rates to be successful.

Kiley is enjoying her work in the supermarket. Today, while she was working in the section of the supermarket that has nuts and other bulk items, a customer needed her help. This customer was trying to figure out a couple of different prices for almonds and cashews. The customer had measured out three pounds of almonds. When she weighed the almonds and printed her price ticket, the price read “$8.97.”

“How much are these per pound?” the customer asked Kiley.

Kiley looked at the bin, but the label had become worn and she could not see the actual ticket. For Kiley to figure this out, she is going to have to use her arithmetic skills. How much are the almonds per pound?

To complete this dilemma, you will need to understand rates and equivalent rates. Pay close attention and we’ll revisit this problem at the end of the Concept.

Guidance

In the world around us there are many times when we need to use a rate. We use rates when we think about how many miles a car can travel on a gallon of gasoline. We use a rate when we think about how fast or slow something or someone goes—that is a rate of travel, commonly called speed. You may be familiar with many different rates, but that doesn’t help us to understand exactly what a rate is. This Concept will explain all about rates.

What is a rate?

A rate is special ratio that represents an amount in terms of a single unit of time or some other quantity. We know that we are working with a rate when we see the key word PER.

The car gets 15 miles per gallon.

Here we are comparing the number of miles to one gallon. This is a rate. It is the rate of miles per gallon of gasoline. Rates can take a different form too. Sometimes, a rate isn’t compared to one, but it is still a rate.

John ran three miles in twenty-one minutes.

What is being compared here? Three miles is being compared to seven minutes. This is the rate. We could use the word “per” in this sentence and it would make perfect sense. When this happens, you know that you are looking at a rate.
How do we write a rate in ratio form?

Once you understand how to identify a rate, you need to know how to write the rate as a ratio since a rate is a special type of ratio.

Let’s look at the dilemma above again.
John ran three miles in twenty-one minutes. To write this as a ratio, we are comparing three miles to twenty-one minutes. The three miles becomes our numerator and the twenty-one minutes our denominator.

\[
\frac{3 \text{ miles}}{21 \text{ minutes}}
\]

The apples are $.99 per pound.

To write this as a ratio it may help to first see that it is a rate. We are comparing the price of apples to the number of pounds. Our key word here is the word “per” and that lets us know that we are comparing to one. Next, we write it as a ratio. Our money amount is our numerator. The number of pounds is our denominator.

\[
\frac{\text{Price of apples}}{\text{number of pounds}} = \frac{.99}{1}
\]

When a rate is compared to one—it is called a unit rate.

Unit rates and rates can be equivalent to each other.

How do we write equivalent rates?
Writing an equivalent rate can be done in a couple of different ways. First, we can take a rate, write it as a ratio and simplify it to a unit rate. Then the two rates will be equivalent.

Karen ran four miles in 20 minutes.
First, we write it as a ratio. We are comparing four miles to twenty minutes.

\[
\frac{20 \text{ minutes}}{4 \text{ miles}}
\]

Next, we simplify the ratio to a unit rate. That means we are comparing to one. We simplify using the greatest common factor of the numerator and the denominator.

\[
\frac{20 \text{ minutes}}{4 \text{ miles}} = \frac{5 \text{ minutes}}{1 \text{ mile}}
\]
These two rates are equivalent. The unit rate is that it took Karen five minutes per mile.
It’s time for you to apply these skills. Write an equivalent rate for each.

Example A

\[ \frac{6 \text{ shirts}}{2 \text{ boxes}} \] how many shirts would there be in six boxes?
Solution: 18 shirts

Example B

How many shirts are there in one box?
Solution: Three shirts in one box

Example C

How many boxes would we need for 24 shirts?
Solution: 8 boxes

Now back to the dilemma with Kiley and the almonds.
The dilemma has to do with the almonds. The customer wanted to know how much they were per pound. She
is looking for the unit rate. Begin by writing a rate that compares three pounds of almonds to the price.

\[ \frac{8.97}{3 \text{ pounds}} \]

Next, we need to figure out the cost of one pound. We can create an equal fraction.

\[ \frac{8.97}{3 \text{ pounds}} = \frac{?}{1 \text{ pound}} \]

We divided by three to go from three pounds to one pound. We can divide 8.97 by three to get the unit price.

\[ 8.97 \div 3 = 2.99 \]

The almonds cost $2.99 per pound.

Guided Practice

Here is one for you to try on your own.

Ron ate five hot dogs in one minute. Now find a unit rate.
Answer

First, we write a ratio that compares hot dogs to minutes.

\[
\frac{5 \text{ hot dogs}}{1 \text{ minute}}
\]

This is a unit rate because it is compared to one. Next, we write an equivalent ratio to this one. We can do this by multiplying the numerator and the denominator by the same number.

\[
\frac{5 \text{ hot dogs}}{1 \text{ minutes}} = \frac{10 \text{ hot dogs}}{2 \text{ minutes}}
\]

Yes it is! But, we have two equivalent rates here! We have a unit rate and a rate that shows more time and more hot dogs.

Video Review

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/5410
Explore More

Directions: Write a rate that compares the quantities described in each problem.

1. Fourteen apples in two barrels
2. Thirty-two crayons in two boxes
3. Eighteen bottles in three carriers
4. Twenty students on four teams
5. Twenty-five students on five teams
6. Fifty students in two classes
7. Ninety students on three buses
8. Thirteen students ate twenty-six cupcakes
9. Twelve campers in two tents
10. Twenty-four hikers per trail
11. Six campers per tent
12. If there are six campers per tent, how many tents for 18 campers?
13. How many tents for 30 campers?
14. How many campers can you fit if you have 12 tents?
15. Sixty students on five teams
Mark adopted a dog today and is at the pet store looking for dog food. He researched the different brands before coming to the store and now he just needs to decide between two brands. Doggie Delights costs $46 for a 20 pound bag. Nature’s Choice costs $55 for a 25 pound bag. Mark wants to compare the cost of dog food per pound to find the better deal. Which bag should Mark buy?

In this concept, you will learn to write and compare unit rates.

Comparing Unit Rates

Equivalent rates can be used to compare different sets of quantities that have the same value. A rate that compares a quantity to one is called a **unit rate**. The unit rate has a denominator equal to one when written as a fraction. Unit rates can be used to find larger equivalent rates.

Let’s use a unit rate to solve a problem.

Mrs. Harris’ class went apple picking. Each student picked 8 apples. At this rate, how many apples were picked by 7 students?

The problem gives the unit rate of 8 apples per student. Solve by finding the equivalent rate where the number of students is 7.

First, write the unit rate as a fraction.

\[
\frac{8 \text{ apples}}{1 \text{ student}}
\]

Next, multiply the numerator and denominator by 7 to find the equivalent rate for 7 students.
8 apples \times \frac{7}{1 \text{ student}} = \frac{56 \text{ apples}}{7 \text{ students}}

There were 56 apples picked by 7 students.

Now let’s solve to find the unit rate.

Laquita picked 12 peaches in 6 minutes. How many peaches did Laquita pick per minute?

The problem gives the larger rate of 12 peaches per 6 minutes. Simplify the rate so the denominator is equal to 1 to find the unit rate.

First, write the rate as a fraction.

\[
\frac{12 \text{ peaches}}{6 \text{ minutes}}
\]

Next, simplify the denominator to equal 1. Divide the numerator and denominator by 6.

\[
\frac{12 \text{ peaches} \div 6}{6 \text{ minutes} \div 6} = \frac{2 \text{ peaches}}{1 \text{ minute}}
\]

Laquita picked 2 peaches per minute.

**Examples**

**Example 1**

Earlier, you were given a problem about Mark at the pet store.

Mark is trying to make a choice. Doggie Delights costs $46 for a 20 pound bag and Nature’s Choice costs $55 for a 25 pound bag. To find the better deal, Mark must compare the price per pound.

First, find the unit price for each brand.

\[
\text{Doggie Delights} = \frac{546}{20 \text{ pound}} = \frac{46 \div 20}{20 \div 20} = \frac{2.30}{1}
\]

\[
\text{Nature’s Choice} = \frac{55}{25 \text{ pound}} = \frac{55 \div 25}{25 \div 25} = \frac{2.20}{1}
\]

Next, compare the price per pound. Doggie Delights is $2.30 per pound. Nature’s Choice is $2.20 per pound.

\[ 2.30 > 2.20 \]

Nature’s Choice is cheaper than Doggie Delights.

Mark decides to buy Nature’s Choice.

**Example 2**

Find the unit rate for the following problem: Harold cuts 7 lawns in 4 hours. How many lawns does Harold cut per hour?
First, write the rate as a fraction.

\[
\frac{7 \text{ lawns}}{6 \text{ hours}}
\]

Next, simplify the denominator to equal 1. Divide the numerator and denominator by 4.

\[
\frac{7 \text{ lawns} \div 4}{4 \text{ hours} \div 4} = \frac{1.75 \text{ lawns}}{1 \text{ hour}}
\]

Harold cuts 1.75 lawns per hour.

**Example 3**

Find the unit rate for the following problem: 24 buttons on 4 shirts.

First, write the rate as a fraction.

\[
\frac{24 \text{ buttons}}{4 \text{ shirts}}
\]

Next, divide the numerator and denominator by 4.

\[
\frac{24 \text{ buttons} \div 4}{4 \text{ shirts} \div 4} = \frac{6 \text{ buttons}}{1 \text{ shirt}}
\]

The unit rate is 6 buttons per shirt.

**Example 4**

Find the unit rate for the following problem: 4 ice cream cones for 2 people.

First, write the rate as a fraction.

\[
\frac{4 \text{ ice cream cones}}{2 \text{ people}}
\]

Next, divide the numerator and denominator by 2.

\[
\frac{4 \text{ ice cream cones} \div 2}{2 \text{ people} \div 2} = \frac{2 \text{ ice cream cones}}{1 \text{ person}}
\]

The unit rate is 2 ice cream cones per person.
Example 5

Find the unit rate for the following problem: 45 gallons in 3 miles.

First, write the rate as a fraction.

\[
\frac{45 \text{ miles}}{3 \text{ gallons}}
\]

Next, divide the numerator and denominator by 3.

\[
\frac{45 \text{ miles}}{3 \text{ gallons}} \div 3 = \frac{15 \text{ miles}}{1 \text{ gallon}}
\]

The unit rate is 15 miles per gallon.

Review

Find the unit rate.

1. Fourteen apples in two barrels
2. Thirty-two crayons in two boxes
3. Eighteen bottles in three carriers
4. Twenty students on four teams
5. Twenty-five students on five teams
6. Fifty students in two classes
7. Ninety students on three buses
8. Thirteen students ate twenty-six cupcakes
9. Twelve campers in two tents
10. Forty-eight hikers on two trails
11. Sixteen hikers on 4 trails
12. Seventy-two bikes on 6 racks
13. Fifteen backpacks for 5 children
14. Twenty-eight slices of pizza for 7 teenagers
15. $24.00 for 2 people

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.5.
Here you’ll learn to find unit rates.

**Let’s Think About It**

Hannah is a runner on her school’s track team. In order to perform well in races, she trains nearly every day. The running coach requires that all athletes run at least 35 miles per week. So far this week, Hannah ran 3 miles on Sunday, 6 miles on Monday, and 5 miles on Tuesday. At her current rate, will Hannah meet the requirement?

In this concept, you will learn how to work with unit rates.

**Guidance**

A **unit rate** is a comparison of two measurements, one of which has a value of 1.

For example:

The cost for 1 pound of apples is $1.50

This can be written as \( \frac{1}{1.50} \) **pound** \( \frac{\text{dollars}}{} \)

Based on this unit rate, the cost for any number of pounds of apples can be calculated. Given the number of units, the unit rate can be used to calculate a total rate. The unit rate can be calculated by dividing one term of a fraction by the other, and reducing the desired term to 1.

Here is an example.
A cyclist pedaled 36 miles in 2 hours. What is her unit rate? In other words, how many miles did she pedal in 1 hour?

First, write the rate as a fraction. The term that needs to be reduced to 1 is hours in order to get a distance per 1 hour. In this case, the term to be reduced to 1 is the denominator.

\[
\frac{36 \text{ miles}}{2 \text{ hours}}
\]

Next, divide both the numerator and denominator by 2 to get a denominator of 1.

A unit rate must have 1 in one of its terms. It can be either the numerator or the denominator.

\[
\frac{36 ÷ 2}{2 ÷ 2} = \frac{18 \text{ miles}}{1 \text{ hour}}
\]

The answer is 18 miles/hour.

**Guided Practice**

Thomas has the record among all his friends for being the fastest texter. Yesterday, the boys decided to find out just how fast. Irwin gave him a message with 240 characters in it, and set a timer. Thomas completed the text in exactly 3 minutes. What is his texting rate per minute?

First, write a fraction.

\[
\frac{240 \text{ characters}}{3 \text{ minutes}}
\]

Next, recognize that the problem is asking for a rate per minute and reduce the fraction to its lowest terms by reducing the number of minutes to 1.

\[
\frac{240 ÷ 3}{3 ÷ 3} = \frac{80}{1}
\]

The answer is 80 characters per minute.

**Examples**

**Example 1**

Nathaniel’s car gets 45 miles for 3 gallons of gasoline. How many miles does he get per gallon?

First, write a fraction. It does not matter which value is on top.

\[
\frac{45 \text{ miles}}{3 \text{ gallons}}
\]

Next, recognize which value needs to be reduced to 1. In this case, "per gallon" signals that gallons must be reduced to 1. 3 gallons must be divided by 3 to get 1.

\[
\frac{45 ÷ 3}{3 ÷ 3} = \frac{15}{1}
\]

The answer is 15 miles per gallon.

**Example 2**

Cassandra has a part time job after school in her mom’s office. She spent 6 hours this week filing a stack of 850 folders. How many folders did she file each hour?

First, write a fraction showing the relationship between hours and folders. Remember, it does not matter which is in the numerator or which is in the denominator.
Next, reduce to lowest terms based on which value the problem calls for. In this problem, it is per hour.

\[
\frac{6}{850} \div \frac{6}{6} = \frac{1}{142}
\]

The answer is 142 folders per hour.

**Example 3**

Find the unit rate for 1 minute: 18 inches in 2 minutes

First, write a fraction.

\[
\frac{18 \text{ inches}}{2 \text{ minutes}}
\]

Next, reduce to lowest terms.

\[
\frac{18 \div 2}{2 \div 2} = \frac{9}{1}
\]

The answer is 9 inches per minute.

**Follow Up**

Remember Hannah from the school track team?

The coach requires all runners to complete 35 miles per week. So far Hannah has run 4 miles on Sunday, 6 miles on Monday, and 5 miles on Tuesday. At her current rate, will Hannah meet the requirement?

First, recognize that the unit rate (miles per day) needs to be found first in order to determine whether or not Hannah will meet her goal.

Next, add the number of miles together that Hannah has already run.

\[4 + 6 + 5 = 15\]

Hannah has run a total of 15 miles in 3 days this week.

Then, write a fraction that compares the number of miles Hannah has already run to the number of days she ran to get her unit rate per day.

\[
\frac{15 \text{ miles}}{3 \text{ days}}
\]

Next, reduce to lowest terms.

\[
\frac{15 \div 3}{3 \div 3} = \frac{5}{1}
\]

Hannah has run 5 miles per day.

Then, think. If Hannah runs 5 miles per day, how many miles will she run in a week?

\[
\frac{5 \text{ miles}}{1 \text{ day}} \times \frac{7 \text{ days}}{1 \text{ week}} = \frac{35 \text{ miles}}{1 \text{ week}}
\]

The answer is that at her current rate of 5 miles per day, Hannah will run 35 miles per week and will meet the coach’s requirements.

**Video Review**
Explore More

Write a unit rate for each ratio.

1. 2 for $10.00
2. 3 for $15.00
3. 5 gallons for $12.50
4. 16 pounds for $40.00
5. 18 inches for $2.00
6. 5 pounds of blueberries for $20.00
7. 40 miles in 80 minutes
8. 20 miles in 4 hours
9. 10 feet in 2 minutes
10. 12 pounds in 6 weeks
11. 14 pounds for $7.00
12. 18 miles in 3 hours
13. 21 inches of cloth costs $7.00
14. 45 miles on 3 gallons of gasoline
15. 200 miles in four hours

Answers for Explore More Problems

To view the Explore More answers, open this PDF file and look for section 5.5.
Indirect Measurement

Here you’ll learn how to apply your knowledge of similar triangles and proportions to model real-life situations and to find unknown measurements indirectly.

**Indirect Measurement**

An application of similar triangles is to measure lengths indirectly. You can use this method to measure the width of a river or canyon or the height of a tall object. The idea is that you model a situation with similar triangles and then use proportions to find the missing measurement indirectly.

What if you were standing next to a building and wanted to know how tall the building was? How could you use your own height and the length of the shadows cast by you and the building to determine the building’s height?

**Examples**

For Examples 1, 2, and 3, use the following information:

In order to estimate the width of a river, the following technique can be used. Use the diagram.

Place three markers, $O, C,$ and $E$ on the upper bank of the river. $E$ is on the edge of the river and $OC \perp CE$. Go across the river and place a marker, $N$ so that it is collinear with $C$ and $E$. Then, walk along the lower bank of the river and place marker $A$, so that $CN \perp NA$. $OC = 50$ feet, $CE = 30$ feet, $NA = 80$ feet.

**Example 1**

Is $\triangle OCE \sim \triangle ANE$? How do you know?
Yes. \( \angle C \cong \angle N \) because they are both right angles. \( \angle OEC \cong \angle AEN \) because they are vertical angles. This means \( \triangle OCE \sim \triangle ANE \) by the AA Similarity Postulate.

**Example 2**

Is \( OC \parallel NA \)? How do you know?

Since the two triangles are similar, we must have \( \angle EOC \cong \angle EAN \). These are alternate interior angles. When alternate interior angles are congruent then lines are parallel, so \( OC \parallel NA \).

**Example 3**

What is the width of the river? Find \( EN \).

Set up a proportion and solve by cross-multiplying.

\[
\frac{30}{EN} = \frac{50}{80}
\]

\[50(EN) = 2400\]
\[EN = 48\]

The river is 48 feet wide.

**Example 4**

A tree outside Ellie’s building casts a 125 foot shadow. At the same time of day, Ellie casts a 5.5 foot shadow. If Ellie is 4 feet 10 inches tall, how tall is the tree?

To solve, start by drawing a picture. We see that the tree and Ellie are parallel, so the two triangles are similar.

\[
\frac{4\text{ ft}, 10\text{ in}}{x} = \frac{5.5\text{ ft}}{125\text{ ft}}
\]

The measurements need to be in the same units. Change everything into inches and then we can cross multiply.

\[
\frac{58\text{ in}}{x} = \frac{66\text{ in}}{1500\text{ in}}
\]
\[87000 = 66x\]
\[x \approx 1318.18\text{ in or } 109.85\text{ ft}\]
Example 5

Cameron is 5 ft tall and casts a 12 ft shadow. At the same time of day, a nearby building casts a 78 ft shadow. How tall is the building?

To solve, set up a proportion that compares height to shadow length for Cameron and the building. Then solve the equation to find the height of the building. Let \( x \) represent the height of the building.

\[
\frac{5\text{ ft}}{12\text{ ft}} = \frac{x}{78\text{ ft}}
\]

\[
12x = 390
\]

\[
x = 32.5\text{ ft}
\]

The building is 32.5 feet tall.

Review

The technique from the guided practice section was used to measure the distance across the Grand Canyon. Use the picture below and \( OC = 72 \text{ ft}, CE = 65 \text{ ft}, \) and \( NA = 14,400 \text{ ft} \) for problems 1 - 3.

1. Find \( EN \) (the distance across the Grand Canyon).
2. Find \( OE \).
3. Find \( EA \).
4. Mark is 6 ft tall and casts a 15 ft shadow. At the same time of day, a nearby building casts a 30 ft shadow. How tall is the building?
5. Karen and Jeff are standing next to each other. Karen casts a 10 ft shadow and Jeff casts an 8 ft shadow. Who is taller? How do you know?
6. Billy is 5 ft 9 inches tall and Bobby is 6 ft tall. Bobby’s shadow is 13 ft long. How long is Billy’s shadow?
7. Sally and her little brother are walking to school. Sally is 4 ft tall and has a shadow that is 3 ft long. Her little brother’s shadow is 2 ft long. How tall is her little brother?
8. Ryan is outside playing basketball. He is 5 ft tall and at this time of day is casting a 12 ft shadow. The basketball hoop is 10 ft tall. How long is the basketball hoop’s shadow?
9. Jack is standing next to a very tall tree and wonders just how tall it is. He knows that he is 6 ft tall and at this moment his shadow is 8 ft long. He measures the shadow of the tree and finds it is 90 ft. How tall is the tree?
10. Thomas, who is 4 ft 9 inches tall is casting a 6 ft shadow. A nearby building is casting a 42 ft shadow. How tall is the building?
1.9. Indirect Measurement

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.5.

Resources

MEDIA

Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/1345
Geri is getting ready to go through the automatic car wash with her mom. Geri’s mom gives her several dollar bills and tells her to use it to get quarters for the car wash. Geri puts a dollar in the change machine and the machine gives her 4 quarters. She puts another dollar in the machine and it gives her 4 more quarters. Geri continues until she has 26 quarters. As she walks back to the car, she begins to think about what she put into the change machine and what came out.

This is a table to represent the change machine’s input and output.
What rule could Geri write to represent what happened to the input to equal the output?
In this concept, you will learn to evaluate and write function rules for an input-output table.

**Writing Function Rules for Input-Output Tables**

A function is when one variable or term depends on another according to a rule. There is a special relationship between the two variables of the function where each value in the input applies to only one value in the output. These rules are called function rules, because they explain how the function operates. The function rule is the same thing as the expression. Here are some hints for writing function rules:

1. Decipher the pattern of the function by asking, “What happened to the input to get the output?”
2. Write the rule as an expression.

Take a look at the following function rule and determine if it is a rule for the data in the table below.

\[
x + 4
\]

First, substitute the input values in for \(x\) to see if you get the corresponding output value.

\[
x + 4 \\
2 + 4 \\
6
\]

This does not equal the corresponding output value of 5.

Look at the other input values. Each term in the input became the term in the output when 3 was added to it. The rule states that four was added. Therefore, this is not a viable rule.

Here is another function.

\[5x\]

Determine if \(5x\) is a function rule for the data in the table below.
First, substitute the input values in for $x$ to see if you get the corresponding output value.

\[
5x \\
5(20) \\
100
\]

Substitute another input value.

\[
5x \\
5(10) \\
50
\]

\[
5x \\
5(5) \\
25
\]

\[
5x \\
5(1) \\
5
\]

So, yes it is. In this case, each term in the input was multiplied by five to get the term in the output. Therefore this rule does work for this table.

**Examples**

**Example 1**

Earlier, you were given a problem about Geri and the change machine.

Geri knows that there are 4 quarters in a dollar, and that is why she put 6 dollars in the machine and received 24 quarters. How can Geri write this as a function rule?

This is a table to represent the change machine’s input and output.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>
Table 1.5:

<table>
<thead>
<tr>
<th>Input (dollars)</th>
<th>Output (quarters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

First, look at the table and ask yourself, “What happened to x (input) to get y (output)?”

What happened to 5 to get 50? What happened to 6 to get 60 and so forth? If you look carefully, you will see that the input value (x) is multiplied by 10 to get the output value.

Next, use a variable for the input and write the rule.

You can write it as an expression, x(10) or 10x. This is the function rule, 10x.

Then, see if the function rule 10x works for each term in the table by plugging the input into the expression and seeing if it equals the listed output?

\[
10x \\
10(5) \\
50
\]

\[
10x \\
10(6) \\
60
\]

\[
10x \\
10(7) \\
70
\]

\[
10x \\
10(8) \\
80
\]

The answer is yes, this rule works for this table.

Example 2

Write a function rule to represent the data in this table.

Table 1.6:
First, look at the table and ask yourself, “What happened to \( x \) (input) to get \( y \) (output)?”

Here two operations were performed. The input value was multiplied by two and then one was subtracted.

Next, use a variable for the input and write the rule.

\[ 2(x) - 1 \]

The answer or function rule is \( 2x - 1 \).

Then, see if the function rule \( 2x - 1 \) works for each term in the table by plugging the input into the expression and seeing if it equals the listed output.

Substitute the input values in for \( x \) in the function \( 2x - 1 \) to see if you get the results in the output column.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>19</td>
</tr>
</tbody>
</table>

\[
2x - 1 \\
2(3) - 1 \\
6 - 1 \\
5 \text{ (output)}
\]

\[
2x - 1 \\
2(5) - 1 \\
10 - 1 \\
9 \text{ (output)}
\]

\[
2x - 1 \\
2(7) - 1 \\
14 - 1 \\
13 \text{ (output)}
\]

\[
2x - 1 \\
2(8) - 1 \\
16 - 1 \\
15 \text{ (output)}
\]
The answer is correct.

Example 3

Determine whether the following rule makes sense for the input-output table.

Rule: $4x$

| Table 1.7: |
|---|---|
| **Input** | **Output** |
| 2 | 10 |
| 3 | 15 |
| 5 | 25 |
| 6 | 30 |

First, let the input value be the variable $x$.

Next, substitute the input values in the expression for $x$.

\[
4x = 10 \text{ (output)} \\
4(2) = 10 \\
8 \neq 10
\]

The answer is no, this rule does not work for this table.

Example 4

Determine whether the following rule makes sense for the input-output table.

Rule: $2x - 1$

| Table 1.8: |
|---|---|
| **Input** | **Output** |
| 2 | 3 |
| 3 | 5 |
| 4 | 7 |
| 6 | 11 |

First, let the input value be the variable $x$.

Next, substitute the input values in the expression for $x$. 

\[ 2x - 1 = 3 \text{ (output)} \]
\[ 2(2) - 1 = 3 \]
\[ 4 - 1 = 3 \]
\[ 3 = 3 \]

\[ 2x - 1 = 5 \]
\[ 2(3) - 1 = 5 \]
\[ 6 - 1 = 5 \]
\[ 5 = 5 \]

\[ 2x - 1 = 7 \]
\[ 2(4) - 1 = 7 \]
\[ 8 - 1 = 7 \]
\[ 7 = 7 \]

\[ 2x - 1 = 11 \]
\[ 2(6) - 1 = 11 \]
\[ 12 - 1 = 11 \]
\[ 11 = 11 \]

The answer is yes, this rule does work for this table.

**Example 5**

Determine whether the following rule makes sense for the input-output table.

**Rule**: $3x$

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
</tr>
</tbody>
</table>

First, let the input value be the variable $x$.

Next, substitute the input values in the expression for $x$.

\[ 3x = 6 \text{ (output)} \]
\[ 3(2) = 6 \]
\[ 6 = 6 \]
\[ 3x = 9 \]
\[ 3(3) = 9 \]
\[ 9 = 9 \]

\[ 3x = 12 \]
\[ 3(4) = 12 \]
\[ 12 = 12 \]

\[ 3x = 18 \]
\[ 3(6) = 18 \]
\[ 18 = 18 \]

The answer is yes, this rule does work for this table.

**Review**

Evaluate each given function rule to determine if the rule works for the data in the table.

1. \(2x + 2\)

**Table 1.10:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

2. \(3x\)

**Table 1.11:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

3. \(5x + 1\)

**Table 1.12:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
</tr>
</tbody>
</table>
### Table 1.12: (continued)

<table>
<thead>
<tr>
<th>3</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>21</td>
</tr>
</tbody>
</table>

4. \(2x\)

### Table 1.13:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

5. \(3x - 1\)

### Table 1.14:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
</tr>
</tbody>
</table>

6. \(2x + 1\)

### Table 1.15:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

7. \(4x\)

### Table 1.16:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

8. \(6x - 3\)
### Table 1.17:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

9. $2x$

### Table 1.18:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

10. $3x - 3$

### Table 1.19:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

Create a table for each rule.

11. $5x$
12. $6x + 1$
13. $2x - 3$
14. $3x + 3$
15. $4x + 1$

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 12.11.

**Resources**
MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/182128

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/182129
Timothy is planning a surprise birthday party for his sister at the local bowling alley. He has already arranged the food but wants to include bowling as an extra. The bowling alley has told Timothy that bowling shoes cost a flat rate of $2.00 and the cost to bowl is $3.00 per game. He has to figure out the cost for his sister and three friends to bowl and how many games they can bowl. Timothy only has $50.00 to spend on bowling. How can he figure out how many games they can bowl and have enough money to pay the total cost for four people?

In this concept, you will learn to evaluate function rules.

**Function Tables**

A function is a relation such that each member of the domain is paired with one and only one member of the range. A set of ordered pairs \((x, y)\) is a relation. The domain is the set of values made up of the \(x\)-coordinates of the ordered pairs while the range is the set of values made up of the \(y\)-values of the ordered pairs. A function table is an output/input table where the input value is a member of the domain and the output value is a member of the range. The output value is the result of an operation or operations performed on the input value and its value depends upon the input number.

Look at the following function table:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
</tr>
</tbody>
</table>
If \( x \) represents the input value and \( y \) represents the output value, the table can be used to write a function rule. A **function rule** is an expression written in either words or symbols to represent the operation or operations performed on the input number to give the output number. From the above table it is obvious that each output number \( y \) is the result of doubling the input number \( x \). The function rule written using symbols is the equation:

\[
y = 2x
\]

Let’s apply a function rule to complete the following input/output table.

Use the function rule \( y = 3x + 2 \) to complete the table below.

**Table 1.21:**

<table>
<thead>
<tr>
<th>Input (( x ))</th>
<th>Output (( y ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
</tr>
</tbody>
</table>

First, substitute the input value of 2 into the function rule for the variable \( x \).

\[
y = 3x + 2
\]

\[
y = 3(2) + 2
\]

Next, perform the multiplication to clear the parenthesis.

\[
y = 6 + 2
\]

Next, perform the addition on the right side of the equation.

\[
y = 8
\]

The answer is 8.

The output value is 8 when the input value is 2.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.

\[
x = 3
\]

\[
y = 3x + 2
\]

\[
y = 3(3) + 2
\]

\[
y = 9 + 2
\]

\[
y = 11
\]

The answer is 11.
1.11. Evaluating Function Rules

\[ x = 4 \]
\[ y = 3x + 2 \]
\[ y = 3(4) + 2 \]
\[ y = 12 + 2 \]
\[ y = 14 \]

The answer is 14.

\[ x = 5 \]
\[ y = 3x + 2 \]
\[ y = 3(5) + 2 \]
\[ y = 15 + 2 \]
\[ y = 17 \]

The answer is 17.

Then, complete the table by filling in the calculated output numbers.

**Examples**

**Example 1**

Earlier, you were given a problem about Timothy and the surprise birthday party.

He needs to figure out how many games of bowling his sister and her friends can play for $50.00 or less.

First, he must write a function rule to represent the information he has from the bowling alley.

Shoes are a flat rate of $2.00 and each game played costs $3.00. The function rule for the information is: \( y = 3x + 2 \) where \( y \) is the total cost and \( x \) is the number of games played.

Next, create an input/output table.

**Table 1.22:**

<table>
<thead>
<tr>
<th>Input ((x))</th>
<th>Output ((y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5.00</td>
</tr>
<tr>
<td>2</td>
<td>$8.00</td>
</tr>
<tr>
<td>3</td>
<td>$11.00</td>
</tr>
<tr>
<td>4</td>
<td>$14.00</td>
</tr>
</tbody>
</table>

First, substitute the input value of 1 into the function rule for the variable \( x \).  
\[
\begin{align*}
  y &= 3x + 2 \\
  y &= 3(1) + 2
\end{align*}
\]

Next, perform the multiplication to clear the parenthesis.

\[
\begin{align*}
  y &= 3(1) + 2 \\
  y &= 3 + 2
\end{align*}
\]
Next, perform the addition on the right side of the equation.

\[
y = 3 + 2 \\
y = 5
\]

The answer is 5.

The output value is $5.00 for one person to bowl one game.

Next, write the output number in the table.

Then, multiply $5.00 by 4 to determine the cost for four people to bowl one game.

\[
5.00 \times 4 = 20.00
\]

The answer is $20.00.

It will cost $20.00 for his sister and three friends to bowl one game.

First, substitute the input value of 2 into the function rule for the variable ’x.’

\[
y = 3x + 2 \\
y = 3(2) + 2
\]

Next, perform the multiplication to clear the parenthesis.

\[
y = 3(2) + 2 \\
y = 6 + 2
\]

Next, perform the addition on the right side of the equation.

\[
y = 6 + 2 \\
y = 8
\]

The answer is 8.

The output value is $8.00 for one person to bowl two games.

Next, write the output number in the table.

Then, multiply $8.00 by 4 to determine the cost for four people to bowl two games.

\[
8.00 \times 4 = 32.00
\]

The answer is $32.00.

It will cost $32.00 for his sister and three friends to bowl two games.

First, substitute the input value of 3 into the function rule for the variable ’x.’
$y = 3x + 2$

Next, perform the multiplication to clear the parenthesis.

$y = 3(3) + 2$

Next, perform the addition on the right side of the equation.

$y = 9 + 2$

The answer is 11.

The output value is $11.00 for one person to bowl three games.

Next, write the output number in the table.

Then, multiply $11.00 by 4 to determine the cost for four people to bowl three games.

$11.00 \times 4 = 44.00$

The answer is $44.00.

It will cost $44.00 for his sister and three friends to bowl three games.

First, substitute the input value of 4 into the function rule for the variable ‘x.’

$y = 3x + 2$

Next, perform the multiplication to clear the parenthesis.

$y = 3(4) + 2$

Next, perform the addition on the right side of the equation.

$y = 12 + 2$

The answer is 14.

The output value is $14.00 for one person to bowl four games.

Next, write the output number in the table.
Then, multiply $14.00 by 4 to determine the cost for four people to bowl four games.

$$14.00 \times 4 = 56.00$$

The answer is $56.00.
It will cost $56.00 for his sister and three friends to bowl four games.
Timothy has enough money for his sister and three friends to bowl three games.

**Examples 2**

Use the following function rule to complete the input/output table.

$$y = 2x - 5$$

<table>
<thead>
<tr>
<th>Input (x)</th>
<th>Output (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>-15</td>
</tr>
<tr>
<td>-1</td>
<td>-7</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

First, substitute the input value of -5 into the function rule for the variable ’x.’

$$y = 2(-5) - 5$$

Next, perform the multiplication to clear the parenthesis.

$$y = -10 - 5$$

Next, perform the addition of the two negative values on the right side of the equation.

$$y = -15$$

The answer is -15.
The output value is -15 when the input value is -5.
Then, write the output number in the table.
Repeat the above process for each of the given input numbers.
1.1. Evaluating Function Rules

\[
x = -1 \\
y = 2x - 5 \\
y = 2(-1) - 5 \\
y = -2 - 5 \\
y = -7
\]

The answer is -7.

\[
x = 2 \\
y = 2x - 5 \\
y = 2(2) - 5 \\
y = 4 - 5 \\
y = -1
\]

The answer is -1.

\[
x = 6 \\
y = 2x - 5 \\
y = 2(6) - 5 \\
y = 12 - 5 \\
y = 7
\]

The answer is 7.

**Example 3**

Use the given function rule to complete the input/output table:

\[y = 4x - 3\]

**Table 1.24:**

<table>
<thead>
<tr>
<th>Input (x)</th>
<th>Output (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
</tr>
<tr>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>9</td>
<td>33</td>
</tr>
</tbody>
</table>

First, substitute the input value of 4 into the function rule for the variable 'x'.

\[
y = 4x - 3 \\
y = 4(4) - 3
\]

Next, perform the multiplication to clear the parenthesis.
\begin{align*}
y & = 4(4) - 3 \\
y & = 16 - 3
\end{align*}

Next, perform the subtraction on the right side of the equation.

\begin{align*}
y & = 16 - 3 \\
y & = 13
\end{align*}

The answer is 13.

The output value is 13 when the input value is 4.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.

\begin{align*}
x & = 5 \\
y & = 4x - 3 \\
y & = 4(5) - 3 \\
y & = 20 - 3 \\
y & = 17
\end{align*}

The answer is 17.

\begin{align*}
x & = 7 \\
y & = 4x - 3 \\
y & = 4(7) - 3 \\
y & = 28 - 3 \\
y & = 25
\end{align*}

The answer is 25.

\begin{align*}
x & = 9 \\
y & = 4x - 3 \\
y & = 4(9) - 3 \\
y & = 36 - 3 \\
y & = 33
\end{align*}

The answer is 33.

**Example 4**

Use the given function rule to complete the input/output table:

\[ y = -2x + 7 \]
First, substitute the input value of -14 into the function rule for the variable 'x.'

\[
y = -2x + 7
\]

\[
y = -2(-14) + 7
\]

Next, perform the multiplication to clear the parenthesis.

\[
y = -2(-14) + 7
\]

\[
y = 28 + 7
\]

Next, perform the addition on the right side of the equation.

\[
y = 28 + 7
\]

\[
y = 35
\]

The answer is 35.

The output value is 35 when the input value is -14.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.

\[
x = -9
\]

\[
y = -2x + 7
\]

\[
y = -2(-9) + 7
\]

\[
y = 18 + 7
\]

\[
y = 25
\]

The answer is 25.

\[
x = 4
\]

\[
y = -2x + 7
\]

\[
y = -2(4) + 7
\]

\[
y = -8 + 7
\]

\[
y = -1
\]

The answer is -1.
\[
\begin{align*}
x &= 11 \\
y &= -2x + 7 \\
y &= -2(11) + 7 \\
y &= -22 + 7 \\
y &= -15
\end{align*}
\]

The answer is -15.

**Example 5**

Use the given function rule to complete the input/output table:

\[
y = 3x - 14
\]

**Table 1.26:**

<table>
<thead>
<tr>
<th>Input ((x))</th>
<th>Output ((y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>-44</td>
</tr>
<tr>
<td>-7</td>
<td>-35</td>
</tr>
<tr>
<td>2</td>
<td>-8</td>
</tr>
<tr>
<td>9</td>
<td>13</td>
</tr>
</tbody>
</table>

First, substitute the input value of -10 into the function rule for the variable \(x\).

\[
y = 3x - 14 \\
y = 3(-10) - 14
\]

Next, perform the multiplication to clear the parenthesis.

\[
y = 3(-10) - 14 \\
y = -30 - 14
\]

Next, perform the addition on the right side of the equation.

\[
y = -30 - 14 \\
y = -44
\]

The answer is -44.

The output value is -44 when the input value is -10.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.
1.11. Evaluating Function Rules

\[
x = -7 \\
y = 3x - 14 \\
y = 3(-7) - 14 \\
y = -21 - 14 \\
y = -35
\]

The answer is -35.

\[
x = 2 \\
y = 3x - 14 \\
y = 3(2) - 14 \\
y = 6 - 14 \\
y = -8
\]

The answer is -8.

\[
x = 9 \\
y = 3x - 14 \\
y = 3(9) - 14 \\
y = 27 - 14 \\
y = 13
\]

The answer is 13.

**Review**

For numbers 1-5, find each output if the function rule is \(y = 3x + 2\).

**Table 1.27:**

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>

For numbers 6-10, find each output if the function rule is \(y = 4x\).

**Table 1.28:**

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
For numbers 11-15, find each output if the function rule is \( y = -3x \).

### Table 1.29:

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>12.</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>14.</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>15.</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Answer each question about functions.

16. A pastry chef needs to purchase enough dough for her cookies. She buys one pound of dough for every twenty cookies she is going to make. She uses the function \( d(c) = \frac{c}{20} \) where \( c \) is the number of cookies and \( d \) is the pounds of dough she should buy. Identify which variable is the domain and which is the range.

17. Evaluate the function \( f(x) = 2x + 7 \) when the domain is \( \{-3, -1, 1, 3\} \).

18. Evaluate the function \( f(x) = \frac{2}{3}x - 6 \) when the domain is \( \{-10, -5, 0, 5, 10\} \).

19. Evaluate the function \( f(x) = 3x - 1 \) when the domain is \( \{5, 6, 7, 8, 9\} \).

20. Evaluate the function \( f(x) = x - 9 \) when the domain is \( \{1, 2, 3, 4, 5\} \).

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 9.2.
Here you’ll learn the meaning of rate of change, how to calculate it, and how to make predictions about the future based on it. Suppose a new gym opened, and it had 20 members after 1 week, 40 members after 2 weeks, and 60 members after 3 weeks. Could you calculate the rate of change in the number of gym members? How is this different than the slope? If this rate continues, how long will it take for the gym to have 300 members?

**Finding the Rate of Change**

When finding the slope of real-world situations, it is often referred to as **rate of change**. “Rate of change” means the same as “slope.” If you are asked to find the rate of change, use the slope formula or make a slope triangle.

**Let’s use rate of change to solve the following problems:**

1. Andrea has a part-time job at the local grocery store. She saves for her vacation at a rate of $15 every week. What is the rate of change in the amount of money Andrea has?

   Begin by finding two ordered pairs. You can make a chart or use the Substitution Property to find two coordinates.

   Using the points (2, 30) and (10, 150). Since (2, 30) is written first, it can be considered the first point, \((x_1, y_1)\). That means \((10, 150) = (x_2, y_2)\). (Note that it doesn’t matter at all which point is considered first, as the slope will end up the same either way.)

   Use the formula: 
   \[
   \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{150 - 30}{10 - 2} = \frac{120}{8} = 15
   \]

   Andrea’s rate of change is \(\frac{\$15}{1\text{ week}}\).

2. A candle has a starting length of 10 inches. Thirty minutes after lighting it, the length is 7 inches. Determine the rate of change in the length of the candle as it burns. How long does it take for the candle to completely burn to nothing?

   Begin by finding two ordered pairs. The candle begins at 10 inches in length. So at time “zero”, the length is 10 inches. The ordered pair representing this is (0, 10). 30 minutes later, the candle is 7 inches, so the ordered pair is (30, 7). Since (0, 10) is written first, it can be called \((x_1, y_1)\). That means \((30, 7) = (x_2, y_2)\).

   Use the formula: 
   \[
   \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 10}{30 - 0} = \frac{-3}{30} = -\frac{1}{10}
   \]

   The candle has a rate of change of -1 inch/10 minutes (the rate is negative because the candle is getting shorter over time). To find the length of time it will take for the candle to burn out, you can create a graph, use guess and check, or solve an equation.
You can create a graph to help visualize the situation. By plotting the ordered pairs you were given and by drawing a straight line connecting them, you can estimate it will take 100 minutes for the candle to burn out.

3. Examine the following graph. It represents a journey made by a large delivery truck on a particular day. During the day, the truck made two deliveries, each one taking one hour. The driver also took a one-hour break for lunch. Identify what is happening at each stage of the journey (stages A through E).

Here is the driver’s journey.

A. The truck sets off and travels 80 miles in 2 hours.
B. The truck covers no distance for 1 hour.
C. The truck covers \((120 - 80) = 40\) miles in 1 hour.
D. The truck covers no distance for 2 hours.
E. The truck covers 120 miles in 2 hours.

To identify what is happening at each leg of the driver’s journey, you are being asked to find each rate of change.

The rate of change for line segment A can be found using either the formula or the slope triangle. Using the slope triangle, vertical change = 80 and the horizontal change = 2.

\[
\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{80\ \text{miles}}{2\ \text{hours}} = \frac{40\ \text{miles}}{1\ \text{hour}}.
\]
Segments $B$ and $D$ are horizontal lines and each has a slope of zero.

The rate of change for line segment $C$ using the slope formula: Rate of change \( \frac{\Delta y}{\Delta x} = \frac{120 - 80}{4 - 3} \text{ miles} = 40 \text{ miles per hour.} \)

The rate of change for line segment $E$ using the slope formula: Rate of change \( \frac{\Delta y}{\Delta x} = \frac{0 - 120}{8 - 6} \text{ miles} = -60 \text{ miles per hour.} \)

The truck is traveling at negative 60 mph. Another way to say this is that the truck is returning home at a rate of 60 mph.

### Examples

#### Example 1

Earlier, you were told that a new gym had 20 members after 1 week, 40 members after 2 weeks, and 60 members after 3 weeks. What is the rate of change in the number of gym members? If this rate continues, how long will it take for the gym to have 300 members?

We have three points to use to calculate the rate of change: (1, 20), (2, 40), and (3, 60). Since only two points are needed to find the rate of change, you can pick any two points and use the slope formula. Note that this only works when the points represent a constant rate of change or a linear equation.

\[
slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{40 - 20}{2 - 1} = \frac{20}{1} = 20.
\]

Therefore the rate of change is 20 members per week. If you choose two other points out of the three, you will get the same result.

Two determine how long it will take for the gym to have 300 members, you can use proportions:

\[
\frac{20 \text{ members}}{1 \text{ week}} = \frac{300 \text{ members}}{x \text{ weeks}}
\]

Solve for $x$ using cross multiplication:

\[
\frac{20}{1} = \frac{300}{x}
\]

\[
20 \times x = 300 \times 1
\]

\[
20x = 300
\]

\[
20x \div 20 = 300 \div 20
\]

\[
x = 15
\]

It will take 15 weeks for the gym to have 300 members if it continues at a rate of adding 20 members per week.
Example 2

Adel spent $125 on groceries in one week. Use this to predict how much Adel will spend per month on groceries, if she keeps buying them at the same rate.

Adel’s weekly rate is $125 per week. Since a month is about 4 weeks, multiply $125/week times 4 weeks:

$$\frac{$125}{1 \text{ week}} \cdot 4 \text{ weeks} = \frac{$125}{1 \text{ week}} \cdot 4 = $500$$

Adel will spend about $500 on groceries per month.

Review

1. How is slope related to rate of change? In what ways is it different?

2. The graph below is a distance-time graph for Mark’s 3.5-mile cycle ride to school. During this particular ride, he rode on cycle paths but the terrain was hilly. His speed varied depending upon the steepness of the hills. He stopped once at a traffic light and at one point he stopped to mend a tire puncture. Identify each section of the graph accordingly.

3. Four hours after she left home, Sheila had traveled 145 miles. Three hours later she had traveled 300 miles. What was her rate of change?

4. Jenna saves $60 every $2 \frac{1}{2}$ weeks. What is the rate of change of her savings?

5. Geoffrey has a rate of change of $\frac{10 \text{ feet}}{\text{second}}$. Write a situation that could fit this slope.

Quick Quiz

1. Find the intercepts of $3x + 6y = 25$ and graph the equation.
2. Find the slope between $(8, 5)$ and $(-5, 6)$.
3. Graph $f(x) = 2x + 1$
4. Graph the ordered pair with the following directions: 4 units west and 6 units north of the origin.
5. Using the graph below, list two “trends” about this data. A trend is something you can conclude about the given data.
Review (Answers)

To see the Review answers, open this **PDF file** and look for section 4.8.
Here you’ll learn how to form equations involving percentages in order to solve problems.

Suppose that there was an article in your school newspaper that said that 80% of the students in your school plan on attending the prom. It also said that 500 students in your school plan on attending the prom. Would you be able to tell from this information how many students there are in your school?

**Percent Equations**

Now that you remember how to convert between decimals and percents, you are ready for the **percent equation**:

\[
\text{part} = \% \text{ rate} \times \text{base}
\]

The key words in a percent equation will help you translate it into a correct algebraic equation. Remember the equal sign symbolizes the word “is” and the multiplication symbol symbolizes the word “of”.

**Let’s solve the following percent problems:**

1. Find 30% of 85.

   You are asked to find the part of 85 that is 30%. First, translate into an equation:

   \[
   n = 30\% \times 85
   \]

   Convert the percent to a decimal and simplify:

   \[
   n = 0.30 \times 85
   \]

   \[
   n = 25.5
   \]

2. A dime is worth what percent of a dollar?

   Since a dime is 10 cents and a dollar is 100 cents, we can set up the following equation:

   \[
   \frac{10}{100} = 10\%
   \]

3. 50 is 15% of what number?

   Translate into an equation:

   \[
   50 = 15\% \times w
   \]

   Rewrite the percent as a decimal and solve:
1.13. Percent Equations

\[ 50 = 0.15 \times w \]

\[ \frac{50}{0.15} = \frac{0.15 \times w}{0.15} \]

\[ 333 \frac{1}{3} = w \]

**Examples**

**Example 1**

Earlier, you were asked how many students attend your school if 80% of the students are attending prom and there are 500 students attending prom.

For the percent equation, this problem gives us the rate and the part. We need to solve for the base, call it \( x \).

The percent equation is:

\[ 500 = 80\% \times x \]

Using the fractional form of the percentage, we can solve this equation for \( x \):

\[ \frac{500}{80} = \frac{80}{100} \times x \]

\[ \frac{100}{80} \times 500 = \frac{100}{80} \times \frac{80}{100} \times x \]

\[ \frac{50000}{80} = x \]

\[ 625 = x \]
Your school has 625 students.

**Example 2**

6 is 2% of what number?

First, use the percent equation:

\[ 6 = 2\% \times n \]

Like Example 1, we should use the fractional form of a percentage. Substitute in \( \frac{2}{100} \) for 2%, since they are equivalent expressions:

\[
\frac{6}{2} = \frac{2}{100} \times n \\
\frac{100}{2} \times 6 = \frac{2}{100} \times \frac{2}{100} \times n \\
\frac{600}{2} = n \\
300 = n
\]

6 is 2% of 300.

**Review**

Answer the following.

1. 32% of 600 is what number?
2. 50% of $9.00 is what number?
3. \( \frac{3}{4} \) of 16 is what number?
4. 9.2% of 500 is what number
5. 8 is 20% of what number?
6. 99 is 180% of what number?
7. What percent of 7.2 is 45?
8. What percent of 150 is 5?
9. What percent of 50 is 2500?
10. $3.50 is 25% of what number?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 3.10.
1.14 Use the Percent Equation to Find Part a

In this concept, you will learn to use the percent equation to find part a.

Finding Part a

You can use the proportion \( \frac{a}{b} = \frac{p}{100} \) to solve for a percent. You can also solve percent problems by using an equation. In this concept, you will use a proportion to create a different kind of equation that will help you solve percent problems.

When you solve the proportion \( \frac{a}{b} = \frac{p}{100} \), you cross multiply to find the missing variable. You can rearrange this formula so that you are solving for just the variable \( a \).

\[
\begin{align*}
\frac{a}{b} &= \frac{p}{100} \\
100a &= pb \\
\frac{100a}{100} &= \frac{pb}{100} \\
a &= \frac{pb}{100}
\end{align*}
\]

You could also say that \( a = \frac{pb}{100} \) is equal to \( a = 0.01pb \). As well, you could convert your percent directly into a decimal and therefore the formula becomes even simpler.

Let’s look at an example.

What is 85% of 90?
First, change the 85% into a decimal. You know that percent means that the denominator of the fraction is 100. Therefore, $85\% = 0.85$.

Next, multiply using the percent equation.

\[ a = 0.01pb \\ a = 0.85 \times 90 \\ a = 76.5 \]

The answer is 76.5.

Let's look at another example.

What is 7% of 900?

First, change the 7% into a decimal. You know that percent means that the denominator of the fraction is 100. Therefore, $7\% = 0.07$.

Next, multiply using the percent equation.

\[ a = 0.01pb \\ a = 0.07 \times 900 \\ a = 63 \]

The answer is 63.

**Examples**

**Example 1**

Earlier, you were given a problem about Jordan and his partial spending money.

Jordan has $85 from his birthday but cannot spend any more than 28% of it on his new computer game. How much can he spend?

First, change the 28% into a decimal.

\[ 28\% = 0.28 \]

Next, multiply using the percent equation.

\[ a = 0.01pb \\ a = 0.28 \times 85 \\ a = 23.8 \]

The answer is 23.8.

Therefore, Jordan can spend up to $23.80 on his new computer game.

**Example 2**

What is 19% of 300?
1.14. Use the Percent Equation to Find Part a

First, change the 19% into a decimal.
19% = 0.19

Next, multiply using the percent equation.

\[ a = 0.01pb \]
\[ a = 0.19 \times 300 \]
\[ a = 57 \]

The answer is 57.

Example 3

What is 22% of 100?
First, change the 22% into a decimal.

\[ 22\% = 0.22 \]

Next, multiply using the percent equation.

\[ a = 0.01pb \]
\[ a = 0.22 \times 100 \]
\[ a = 22 \]

The answer is 22.

Example 4

What is 8% of 57?
First, change the 8% into a decimal.

\[ 8\% = 0.08 \]

Next, multiply using the percent equation.

\[ a = 0.01pb \]
\[ a = 0.08 \times 57 \]
\[ a = 4.56 \]

The answer is 4.56.
Example 5

What is 17% of 80?
First, change the 17% into a decimal.

\[ 17\% = 0.17 \]

Next, multiply using the percent equation.

\[ a = 0.01pb \]
\[ a = 0.17 \times 80 \]
\[ a = 13.6 \]

The answer is 13.6.

Review

Solve each percent problem. Round your answers to the nearest tenth when necessary.

1. How much is 15% of 73?
2. What is 70% of 5?
3. What is 3% of 4 million?
4. What is 18% of 30?
5. What is 22% of 56?
6. What is 19% of 300?
7. What is 21% of 45?
8. What is 34% of 250?
9. What is 33% of 675?
10. What is 3% of 700?
11. What is 11% of 955?
12. What is 14% of 55?
13. What is 37% of 17?
14. What is 20% of 9?
15. What is 2% of 18?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.7.

Resources
1.15 Use the Percent Equation to Find the Percent

In this concept, you will learn to use the percent equation to find percent.

Donna is framing a picture with a blue border. Her picture has an area of 400 square inches. After framing with the border, Donna wants her final framed picture to be 1200 square inches. What percent of the final size is her original picture?

In this concept, you will use the percent equation to find a percent.

**Percent Equation**

You can use the proportion \( \frac{a}{b} = \frac{p}{100} \) to solve for a percent. You can also solve percent problems by using an equation. In this concept, you will use a proportion to create a different kind of equation that will help you solve percent problems.

To find percent, you know that the number being compared \((a)\) to the base \((b)\) is equal to the percent \((p)\). Therefore the equation to use is:

\[
a = \frac{P}{100} \times b
\]

Let’s look at a problem.

What percent of 32 is 18?

First, you know that you are looking for a percent. You want to set up the percent equation to solve for \(p\).

\[
18 = \frac{P}{100} \times 32
\]
Next, solve for the value of $\frac{p}{100}$ by dividing both sides by 32.

$$\frac{18}{\frac{32p}{100}} = \frac{18}{32} \Rightarrow \frac{p}{100} = 0.5625$$

Then, solve for $p$, the percent by multiplying both sides by 100.

$$100 \times \frac{p}{100} = 0.5625 \times 100 \Rightarrow p = 56.25$$

The answer is 56.25%.

Therefore, 18 is 56.25% of 32.

Let’s try another example.

10 is what percent of 12?

First, set up the percent equation to solve for $p$.

$$10 = \frac{p}{100} \times 12$$

Next, solve for the value of $\frac{p}{100}$ by dividing both sides by 12.

$$\frac{10}{\frac{12p}{100}} = \frac{10}{12} \Rightarrow \frac{p}{100} = 0.833$$

Then, solve for $p$, the percent by multiplying both sides by 100.

$$100 \times \frac{p}{100} = 0.833 \times 100 \Rightarrow p = 83.3$$

The answer is 83.3%.

Therefore, 10 is 83.3% of 12.

Examples

Example 1

Earlier, you were given a problem about Donna and her frame.
Donna’s picture is 400 square inches and the final framed picture is 1200 square inches. She wants to know what percent of the final framed picture is her photo.

First, set up the percent equation to solve for \( p \).

\[
400 = \frac{p}{100} \times 1200
\]

Next, solve for the value of \( \frac{p}{100} \) by dividing both sides by 1200.

\[
\frac{400}{1200} = \frac{p}{100} \times \frac{100}{100} = 0.333
\]

Then, solve for \( p \), the percent by multiplying both sides by 100.

\[
100 \times \frac{p}{100} = 0.333 \times 100
\]

The answer is 33.3%.

Therefore, Donna’s original picture represents 33.3% of the final framed picture.

**Example 2**

33 is what percent of 50?

First, set up the percent equation to solve for \( p \).

\[
33 = \frac{p}{100} \times 50
\]

Next, solve for the value of \( \frac{p}{100} \) by dividing both sides by 50.

\[
\frac{33}{50} = \frac{p}{100}
\]

Then, solve for \( p \), the percent by multiplying both sides by 100.

\[
100 \times \frac{p}{100} = 0.66 \times 100
\]

The answer is 66%.

Therefore, 33 is 66% of 50.
Example 3

18 is what percent of 20?

First, set up the percent equation to solve for $p$.

$$18 = \frac{p}{100} \times 20$$

Next, solve for the value of $\frac{p}{100}$ by dividing both sides by 20.

$$18 = \frac{20p}{100}$$

$$\frac{18}{20} = \frac{p}{100}$$

$$0.9 = \frac{p}{100}$$

Then, solve for $p$, the percent by multiplying both sides by 100.

$$100 \times \frac{p}{100} = 0.9 \times 100$$

$$p = 90$$

The answer is 90%.

Therefore, 18 is 90% of 20.

Example 4

5 is what percent of 300?

First, set up the percent equation to solve for $p$.

$$5 = \frac{p}{100} \times 300$$

Next, solve for the value of $\frac{p}{100}$ by dividing both sides by 300.

$$5 = \frac{300p}{100}$$

$$\frac{5}{300} = \frac{p}{100}$$

$$0.0167 = \frac{p}{100}$$

Then, solve for $p$, the percent by multiplying both sides by 100.

$$100 \times \frac{p}{100} = 0.0167 \times 100$$

$$p = 1.67$$

The answer is 1.67%.

Therefore, 5 is 1.67% of 300.
Example 5

60 is what percent of 400?
First, set up the percent equation to solve for \( p \).

\[
60 = \frac{p}{100} \times 400
\]

Next, solve for the value of \( \frac{p}{100} \) by dividing both sides by 400.

\[
\frac{60}{400} = \frac{p}{100}
\]

Then, solve for \( p \), the percent by multiplying both sides by 100.

\[
100 \times \frac{p}{100} = 0.15 \times 100
\]

The answer is 15%.
Therefore, 60 is 15% of 400.

Review

Solve each percent problem by using the percent equation. You may round when necessary.

1. What percent of 600 is 82?
2. What percent of 18 is 17?
3. 150 is what percent of 175?
4. 200 is what percent of 450?
5. 34 is what percent of 70?
6. 12 is what percent of 88?
7. 15 is what percent of 90?
8. 230 is what percent of 600?
9. 334 is what percent of 1000?
10. 2 is what percent of 8?
11. 55 is what percent of 1800?
12. 61 is what percent of 80?
13. 33 is what percent of 90?
14. 78 is what percent of 156?
15. 19 is what percent of 31?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.8.
1.16 Use the Percent Equation to Find the Base, b

In this concept, you will learn to use the percent equation to find part b.

Alice lives in New Hampshire and by mid-September 35% of the trees lose their leaves. Alice considers the trees in her grandparents’ grove, and sees that approximately 850 trees have lost their leaves. How can Alice know how many trees are there in all?

In this concept, you will learn to use the percent equation to find the base, b.

Finding the Base b

You can use the proportion \( \frac{a}{b} = \frac{p}{100} \) to solve for a percent. You can also solve percent problems by using an equation. In this concept, you will use a proportion to create a different kind of equation that will help you solve percent problems. Sometimes, you will know the percent and a part of the ratio, or part a, but you will need to find the whole or the base, b.

When you solve the proportion \( \frac{a}{b} = \frac{p}{100} \), you cross multiply to find the missing variable. You can rearrange this formula so that you are solving for just the variable a.

\[
\begin{align*}
\frac{a}{b} &= \frac{p}{100} \\
100a &= pb \\
\frac{100a}{100} &= \frac{pb}{100} \\
a &= \frac{pb}{100}
\end{align*}
\]

You could also say that \( a = \frac{pb}{100} \) is equal to \( a = 0.01pb \). As well, you could convert your percent directly into a decimal and therefore the formula becomes even simpler.
Let’s look at a problem.
78 is 65% of what number?
First, write the equation. Remember that 65% is the same as $\frac{65}{100}$.

$$78 = \frac{65}{100} \times b$$

or

$$78 = 0.65 \times b$$

Next, divide both sides of the equation by 0.65 to solve for $b$.

$$\frac{78}{0.65} = \frac{0.65 \times b}{0.65}$$

$$b = 120$$

The answer is 120.
Therefore, 78 is 65% of 120.

Let’s try another example.
11 is 77% of what number?
First, write the equation. Remember that 77% is the same as $\frac{77}{100}$.

$$11 = \frac{77}{100} \times b$$

or

$$11 = 0.77 \times b$$

Next, divide both sides of the equation by 0.77 to solve for $b$.

$$\frac{11}{0.77} = \frac{0.77 \times b}{0.77}$$

$$b = 14.29$$

The answer is 14.29.
Therefore, 11 is 77% of 14.29.

In this problem, you could round to the nearest hundredths place as you did here. Sometimes, you may be asked to round to the nearest tenths place. In that case, the answer would have been 14.3.
Examples

Example 1

Earlier, you were given a problem about Alice’s family grove.

Alice knows that 850 trees in the grove lost their leaves, but doesn’t know the total number of trees in the grove. She also knows that the 850 represents 35% of the total grove.

First, write the equation. Remember that 35% is the same as \( \frac{35}{100} \).

\[
850 = \frac{35}{100} \times b
\]

or

\[
850 = 0.35 \times b
\]

Next, divide both sides of the equation by 0.35 to solve for \( b \).

\[
\frac{850}{0.35} = \frac{0.35 \times b}{0.35}
\]

\[
2428.57 = b
\]

The answer is 2428.

Therefore, 850 is approximately 35% of 2428.

Example 2

25 is 60% of what number?

First, write the equation. Remember that 60% is the same as \( \frac{60}{100} \).

\[
25 = \frac{60}{100} \times b
\]

or

\[
25 = 0.60 \times b
\]

Next, divide both sides of the equation by 0.60 to solve for \( b \).

\[
\frac{25}{0.60} = \frac{0.60 \times b}{0.60}
\]

\[
41.67 = b
\]

The answer is 41.67.

Therefore, 25 is 60% of 41.67.
Example 3

10 is 50% of what number?
First, write the equation. Remember that 50% is the same as \( \frac{50}{100} \).

\[
10 = \frac{50}{100} \times b
\]
or

\[
10 = 0.50 \times b
\]

Next, divide both sides of the equation by 0.50 to solve for \( b \).

\[
\frac{10}{0.50} = \frac{0.50 \times b}{0.50}
\]

\[
b = 20
\]

The answer is 20.
Therefore, 10 is 50% of 20.

Example 4

45 is 20% of what number?
First, write the equation. Remember that 20% is the same as \( \frac{20}{100} \).

\[
45 = \frac{20}{100} \times b
\]
or

\[
45 = 0.20 \times b
\]

Next, divide both sides of the equation by 0.20 to solve for \( b \).

\[
\frac{45}{0.20} = \frac{0.20 \times b}{0.20}
\]

\[
b = 225
\]

The answer is 225.
Therefore, 45 is 20% of 225.
Example 5

68 is 40% of what number?

First, write the equation. Remember that 40% is the same as \( \frac{40}{100} \).

\[
68 = \frac{40}{100} \times b
\]

or

\[
68 = 0.40 \times b
\]

Next, divide both sides of the equation by 0.40 to solve for \( b \).

\[
\frac{68}{0.40} = \frac{0.40 \times b}{0.40}
\]

\[
b = \frac{170}{0.40}
\]

The answer is 170.
Therefore, 68 is 40% of 170.

Review

Solve each percent problem. You may round your answers to the nearest tenth when necessary.

1. 23 is 9% of what number?
2. 10 is 35% of what number?
3. 580 is 82% of what number?
4. 58 is 8% of what number?
5. 58 is 80% of what number?
6. 11 is 82% of what number?
7. 33 is 2% of what number?
8. 14 is 9% of what number?
9. 50 is 67% of what number?
10. 33 is 45% of what number?
11. 40 is 80% of what number?
12. 68 is 99% of what number?
13. 78 is 55% of what number?
14. 16 is 12% of what number?
15. 1450 is 80% of what number?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.9.
Here you’ll learn how to convert or cancel dimensions in order to solve real-world problems.

What if you went to the grocery store and bought 3 gallons of milk? Could you determine how many pints of milk you purchased? Or how about if you bought 16 pints of milk? How many gallons would this be?

**Dimensional Analysis**

Real-world information is given in **dimensions**, or the units in which the value is measured. For example, the following are all examples of dimensions.

- Inches
- Feet
- Liters
- Micrograms
- Acres
- Hours
- Pounds
- Students

Analyzing dimensions can help you solve problems in travel, astronomy, physics, engineering, forensics, and quality. Solving problems by converting dimensions or canceling dimensions is the focus of this Concept.

Consider the distance formula \( distance = rate \cdot time \). This formula can be rewritten for rate. \( rate = \frac{distance}{time} \). If distance is measured in kilometers, and time is measured in hours, the rate would have the dimensions \( \text{kilometers/hour} \).

You can treat dimensions as variables. Identical units can divide out, or cancel. For example, \( \frac{\text{kilometers}}{\text{hour}} \cdot \text{hour} \rightarrow \text{kilometers} \).

Sometimes the units will not divide out. In this case, a **conversion factor** is needed. A conversion factor is a multiplier for converting between one set of units to another.

The process of using units or dimensions to help solve a problem is called **dimensional analysis**. It is very useful in real-world applications as shown in the following problems.
Let's use dimensional analysis to complete the following problems:

1. Convert \( \frac{35 \text{ kilometers}}{\text{hour}} \) to meters.

Since kilometers \( \neq \) meters, you need to convert kilometers to meters to get the answer. You know there are 1,000 meters in a kilometer. Therefore, you will need to multiply the original dimension by this factor.

\[
\frac{35 \text{ kilometers}}{\text{hour}} \cdot \frac{1000 \text{ meters}}{1 \text{ kilometer}} \rightarrow \frac{35 \text{ kilometers}}{\text{hour}} \cdot \frac{1000 \text{ meters}}{35 \text{ kilometers}} = \frac{35(1000) \text{ meters}}{\text{hour}}.
\]

2. How many seconds are in a month?

This situation can be solved easily using multiplication. However, the process you use when multiplying the values together is an example of dimensional analysis.

Begin with what you know:

- 60 seconds in one minute
- 60 minutes in one hour
- 24 hours in one day
- Approximately 30 days in one month

Now write the expression to convert the seconds in one minute to one month.

\[
\frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{30 \text{ days}}{1 \text{ month}}
\]

Identical units cross-cancel.

\[
\frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{30 \text{ days}}{1 \text{ month}} = \frac{60 \cdot 60 \cdot 24 \cdot 30 \text{ seconds}}{1 \cdot 1 \cdot 1 \text{ month}} = 2,592,000 \frac{\text{seconds}}{\text{month}}
\]
3. How many grams are in 5 pounds?

Begin by writing all the conversions you know related to this situation.

\[
\text{1 gram} \approx 0.0353 \text{ ounces}
\]

\[
16 \text{ ounces} = 1 \text{ pound}
\]

Write your dimensional analysis.

\[
5 \text{ pounds} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \cdot \frac{1 \text{ gram}}{0.0353 \text{ ounce}}
\]

Cross-cancel identical units and multiply.

\[
5 \text{ pounds} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \cdot \frac{1 \text{ gram}}{0.0353 \text{ ounce}} = 2226.29 \text{ grams}
\]

**Examples**

**Example 1**

Earlier, you were told to suppose that you bought 3 gallons of milk at the grocery store. How many pints of milk did you purchase? How many gallons are in 16 pints of milk?

1 gallon is approximately 8 pints. You can use this conversion factor to determine how many pints are in 3 gallons of milk:
Cross-cancel and multiply:

\[
\frac{3 \text{ gallons}}{1 \text{ gallon}} \cdot \frac{8 \text{ pints}}{1 \text{ gallon}} = 24 \text{ pints}
\]

You bought 24 pints of milk.

To determine how many gallons are in 16 pints of milk, use the same conversion factor:

\[
\frac{16 \text{ pints}}{8 \text{ pints}} \cdot \frac{1 \text{ gallon}}{8 \text{ pints}} = 2 \text{ gallons}
\]

There are 2 gallons in 16 pints of milk.

**Example 2**

You are traveling in Europe and want to know how fast to drive to maximize fuel efficiency. The optimal driving speed for fuel efficiency is 55 miles per hour. How fast would that be in kilometers per hour?

Since 1 mile is approximately 1.6 kilometers:

\[
\frac{55 \text{ miles}}{\text{hour}} \cdot \frac{1.6 \text{ kilometers}}{1 \text{ mile}} \rightarrow \frac{55 \text{ miles}}{\text{hour}} \cdot \frac{1.6 \text{ kilometers}}{1 \text{ mile}} \rightarrow 88 \text{ kilometers per hour}
\]

The optimal speed for fuel efficiency is 88 kilometers per hour.

**Review**

1. True or false? Dimensional analysis is the study of space and time.
2. By using dimensional analysis, what happens to identical units that appear diagonally in the multiplication of fractions?
3. How many feet are in a mile?
4. How many inches are in a mile?
5. How many seconds are in a day?
6. How many seconds are in a year?
7. How many feet are in a furlong?
8. How many inches are in 100 yards (one football field)?
9. How many centimeters are in 5 inches?
10. How many meters are between first and second base (90 feet)?
11. How many meters are in 16 yards?
12. How many cups are in 6 liters?
13. How many cubic inches make up one ounce?
14. How many milliliters make up 8 ounces?
15. How many grams are in 100 pounds?
16. An allergy pill contains 25 mg of Diphenhydramine. If 1 gram = 15.432 grains, how many grains of this medication are in the allergy pill?
17. A healthy individual’s heart beats about 68 times per minute. How many beats per hour is this?
18. You live 6.2 miles from the grocery store. How many fathoms is this? (6 feet = 1 fathom)
19. The cost of gas in England is 96.4 pound sterling/liter. How much is this in U.S. dollars/gallon? (3.875 litres = 1 gallon and 1.47 US$ = 1 pound sterling)
20. Light travels \(\frac{186,000 \text{ miles}}{\text{second}}\). How long is one light year?
21. Another way to describe light years is in astronomical units. If 1 light year = 63,240 AU (astronomical units), how far in AUs is Alpha Centauri, which is 4.32 light years from the Earth?
22. How many square feet is 16 acres?
23. A person weighs 264 pounds. How many kilograms is this weight?
24. A car is traveling 65 miles/hour and crosses into Canada. What is this speed in km/hr?
25. A large soda cup holds 32 ounces. What is this capacity in cubic inches?
26. A space shuttle travels 28,000 mph. What is this distance in feet/second?
27. How many hours are in a fortnight (two weeks)?
28. How many fortnights (two-week periods) are in 2 years?
29. A semi truck weighs 32,000 pounds empty. How many tons is this weight?
30. Which has the greatest volume: a 2-liter bottle of soda, one gallon of water, or 10 pints of human blood?

Mixed Review

31. Solve for \(x\): \(-2x + 8 = 8(1 - 4x)\).
32. Simplify: \(3 - 2(5 - 8h) + 13h \cdot 3\).
33. Find the difference: \(-26.375 - \left(-14\frac{1}{8}\right)\).
34. Find the product: \(-2\frac{3}{4} \cdot \frac{9}{10}\).
35. Simplify: \(\sqrt{80}\).
36. Is 5.5 an irrational number? Explain your answer.

Use the relation given for the following questions: \(\{(0, 8), (1, 4), (2, 2), (3, 1), (4, \frac{1}{2}), (5, \frac{1}{4})\}\).

37. State the domain.
38. State the range.
39. Is this relation a function? Explain your answer.
40. What seems to be the pattern in this relation?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.13.

Understand ratio concepts and use ratio reasoning to solve problems.

6.RP.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, “The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak.” “For every vote candidate A received, candidate C received nearly three votes.”

6.RP.2 Understand the concept of a unit rate \(\frac{a}{b}\) associated with a ratio \(\frac{a}{b}\) with \(b \neq 0\), and use rate language in the context of a ratio relationship. For example, “This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar.” “We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger.”

6.RP.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
a. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

b. Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.
Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for \((2/3) \div (3/4)\) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that \((2/3) \div (3/4) = 8/9\) because \(3/4\) of \(8/9\) is \(2/3\). (In general, \((a/b) \div (c/d) = ad/bc\).) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?
### 2.1 Mixed Numbers in Applications

Here you’ll multiply mixed numbers and fractions to solve real-world multiplication problems.

Suppose you’re skateboarding at an average rate of $15\frac{1}{2}$ miles per hour, and you skateboard for $\frac{3}{5}$ of an hour. How far did you travel? In this Concept, you’ll learn how to multiply mixed numbers and fractions so that you can answer real-world questions such as this.

**Mixed Numbers in Applications**

You’ve decided to make cookies for a party. The recipe you’ve chosen makes 6 dozen cookies, but you only need 2 dozen. How do you reduce the recipe?

In this case, you should not use subtraction to find the new values. Subtraction means to make less by taking away. You haven’t made any cookies; therefore, you cannot take any away. Instead, you need to make $\frac{2}{6}$ or $\frac{1}{3}$ of the original recipe. This process involves multiplying fractions.

For any real numbers $a, b, c,$ and $d$, where $b \neq 0$ and $d \neq 0$:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

**MEDIA**

Click image to the left or use the URL below.

URL: [https://www.ck12.org/flx/render/embeddedobject/79458](https://www.ck12.org/flx/render/embeddedobject/79458)
The original cookie recipe calls for 8 cups flour. How much is needed for the reduced recipe?

Begin by writing the multiplication situation. \(8 \cdot \frac{1}{3}\). You need to rewrite this product in the form of the property above. In order to perform this multiplication, you need to rewrite 8 as the fraction \(\frac{8}{1}\).

\[
8 \times \frac{1}{3} = \frac{8}{1} \times \frac{1}{3} = \frac{8 \cdot 1}{1 \cdot 3} = \frac{8}{3} = 2\frac{2}{3}
\]

You will need \(2\frac{2}{3}\) cups of flour.

**Multiplication Properties**

Properties that hold true for addition such as the Associative Property and Commutative Property also hold true for multiplication. They are summarized below.

The **Associative Property of Multiplication**: For any real numbers \(a, b,\) and \(c,\)

\[
(a \cdot b) \cdot c = a \cdot (b \cdot c)
\]

The **Commutative Property of Multiplication**: For any real numbers \(a\) and \(b,\)

\[
a(b) = b(a)
\]

The **Same Sign Multiplication Rule**: The product of two positive or two negative numbers is positive.

The **Different Sign Multiplication Rule**: The product of a positive number and a negative number is a negative number.

**Ayinde is making a dog house that is**

Since the formula for area is

\[
area = length \times width,
\]

we plug in the values for length and width:

\[
area = 3\frac{1}{2} \times 2\frac{2}{3}.
\]

We first need to turn the mixed fractions into improper fractions:
area = \(3 \frac{1}{2} \times 2 \frac{2}{3} = \frac{7}{2} \times \frac{8}{3} = \frac{7 \times 8}{2 \times 3} = \frac{56}{6}\).

Now we turn the improper fraction back into a mixed fraction. Since 56 divided by 6 is 9 with a remainder of 2, we get:

\[
\frac{56}{6} = 9 \frac{2}{6} = 9 \frac{1}{3}
\]

The dog house will have an area of 9\(\frac{1}{3}\) square feet.

**Note:** The units of the area are square feet, or feet squared, because we multiplied two numbers each with units of feet:

feet \times feet = feet^2,

which we call square feet.

**Doris’s truck gets**

Begin by writing each mixed number as an improper fraction.

\[
10 \frac{2}{3} = \frac{32}{3} \quad \quad \quad \quad 5 \frac{1}{2} = \frac{11}{2}
\]

Now multiply the two values together.

\[
\frac{32}{3} \cdot \frac{11}{2} = \frac{352}{6} = \frac{58 \frac{4}{6}}{6} = \frac{58 \frac{2}{3}}{3}
\]

Doris can travel 58 \(\frac{2}{3}\) miles on 5.5 gallons of gas.

**Example**

Anne has a bar of chocolate and she offers Bill a piece. Bill quickly breaks off \(\frac{1}{4}\) of the bar and eats it. Another friend, Cindy, takes \(\frac{1}{3}\) of what was left. Anne splits the remaining candy bar into two equal pieces, which she shares with a third friend, Dora. How much of the candy bar does each person get?

Think of the bar as one whole.

\[1 - \frac{1}{4} = \frac{3}{4}\]. This is the amount remaining after Bill takes his piece.
\[ \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}. \] This is the fraction Cindy receives.

\[ \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}. \] This is the amount remaining after Cindy takes her piece.

Anne divides the remaining bar into two equal pieces. Every person receives \( \frac{1}{4} \) of the bar.

**Review**

Multiply the following rational numbers.

1. \( \frac{3}{4} \times \frac{1}{3} \)
2. \( \frac{15}{11} \times \frac{9}{7} \)
3. \( \frac{2}{7} \times 3.5 \)
4. \( \frac{1}{12} \times \frac{1}{14} \)
5. \( \frac{7}{27} \times \frac{11}{14} \)
6. \( \left( \frac{2}{3} \right)^2 \)
7. \( \frac{1}{11} \times \frac{22}{21} \times \frac{7}{10} \)
8. \( 5.75 \cdot 0 \)

In 9 - 11, state the property that applies to each of the following situations.

9. A gardener is planting vegetables for the coming growing season. He wishes to plant potatoes and has a choice of a single 8 by 7 meter plot, or two smaller plots of 3 by 7 meters and 5 by 7 meters. Which option gives him the largest area for his potatoes?

10. Andrew is counting his money. He puts all his money into $10 piles. He has one pile. How much money does Andrew have?

11. Nadia and Peter are raising money by washing cars. Nadia is charging $3 per car, and she washes five cars in the first morning. Peter charges $5 per car (including a wax). In the first morning, he washes and waxes three cars. Who has raised the most money?

12. Teo is making a flower box that is 5-and-a-half inches by 15-and-a-half inches. How many square inches will he have in which to plant flowers?

**Mixed Review**

13. Compare these rational numbers: \( \frac{16}{27} \text{ and } \frac{2}{3} \).


15. Give an example of a proper fraction. How is this different from an improper fraction?

16. Which property is being applied? \( 16 - (-14) = 16 + 14 = 30 \)

17. Simplify \( 11 \frac{1}{2} + \frac{2}{5} \).

**Answers for Review Problems**

To see the Review answers, open this PDF file and look for section 2.7.
Rational Numbers in Applications

Here you’ll apply the properties of addition and subtraction to solve real-world problems involving rational numbers.

Rational Numbers in Applications

Let’s use the skills we learned in the last concept to solve some real-world problems.

Real-World Application: School Trip

Peter is hoping to travel on a school trip to Europe. The ticket costs $2400. Peter has several relatives who have pledged to help him with the ticket cost. His parents have told him that they will cover half the cost. His grandma Zenoviea will pay one sixth, and his grandparents in Florida will send him one fourth of the cost. What fraction of the cost can Peter count on his relatives to provide?

The first thing we need to do is extract the relevant information. Peter’s parents will provide $\frac{1}{2}$ the cost; his grandma Zenoviea will provide $\frac{1}{6}$; and his grandparents in Florida $\frac{1}{4}$. We need to find the sum of those numbers, or $\frac{1}{2} + \frac{1}{6} + \frac{1}{4}$.

To determine the sum, we first need to find the LCD. The LCM of 2, 6 and 4 is 12, so that’s our LCD. Now we can find equivalent fractions:

$$
\frac{1}{2} = \frac{6 \cdot 1}{6 \cdot 2} = \frac{6}{12} \\
\frac{1}{6} = \frac{2 \cdot 1}{2 \cdot 6} = \frac{2}{12} \\
\frac{1}{4} = \frac{3 \cdot 1}{3 \cdot 4} = \frac{3}{12}
$$

Putting them all together: $\frac{6}{12} + \frac{2}{12} + \frac{3}{12} = \frac{11}{12}$.

Peter will get $\frac{11}{12}$ the cost of the trip, or $2200 out of $2400, from his family.

Real-World Application: Property Management

A property management firm is buying parcels of land in order to build a small community of condominiums. It has just bought three adjacent plots of land. The first is four-fifths of an acre, the second is five-twelfths of an acre, and...
the third is nineteen-twentieths of an acre. The firm knows that it must allow one-sixth of an acre for utilities and a small access road. How much of the remaining land is available for development?

The first thing we need to do is extract the relevant information. The plots of land measure \(\frac{4}{5}, \frac{5}{12},\) and \(\frac{19}{20}\) acres, and the firm can use all of that land except for \(\frac{1}{6}\) of an acre. The total amount of land the firm can use is therefore \(\frac{4}{5} + \frac{5}{12} + \frac{19}{20} - \frac{1}{6}\) acres.

We can add and subtract multiple fractions at once just by finding a common denominator for all of them. The factors of 5, 9, 20, and 6 are as follows:

\[
\begin{align*}
5 & = 5 \\
12 & = 2 \cdot 2 \cdot 3 \\
20 & = 2 \cdot 2 \cdot 5 \\
6 & = 2 \cdot 3
\end{align*}
\]

We need a 5, two 2’s, and a 3 in our LCD. \(2 \cdot 2 \cdot 3 \cdot 5 = 60\), so that’s our common denominator. Now to convert the fractions:

\[
\begin{align*}
\frac{4}{5} &= \frac{12 \cdot 4}{12 \cdot 5} = \frac{48}{60} \\
\frac{5}{12} &= \frac{5 \cdot 5}{5 \cdot 12} = \frac{25}{60} \\
\frac{5}{12} &= \frac{5 \cdot 5}{5 \cdot 12} = \frac{25}{60} \\
\frac{19}{20} &= \frac{3 \cdot 19}{3 \cdot 20} = \frac{57}{60} \\
\frac{1}{6} &= \frac{10 \cdot 1}{10 \cdot 6} = \frac{10}{60}
\end{align*}
\]

We can rewrite our sum as \(\frac{48}{60} + \frac{25}{60} + \frac{57}{60} - \frac{10}{60} = \frac{48 + 25 + 57 - 10}{60} = \frac{120}{60}\).

Next, we need to reduce this fraction. We can see immediately that the numerator is twice the denominator, so this fraction reduces to \(\frac{2}{1}\) or simply 2. One is sometimes called the invisible denominator, because every whole number can be thought of as a rational number whose denominator is one.

The property firm has two acres available for development.

**Evaluate Change Using a Variable Expression**

When we write algebraic expressions to represent a real quantity, the difference between two values is the change in that quantity.

\[
\text{Intensity} = \frac{3}{d^2}
\]

The intensity of light hitting a detector when it is held a certain distance from a bulb is given by this equation:
where \(d\) is the distance measured in \textit{meters}, and intensity is measured in \textit{lumens}. Calculate the change in intensity when the detector is moved from two meters to three meters away.

We first find the values of the intensity at distances of two and three meters.

\[
\text{Intensity (2)} = \frac{3}{(2)^2} = \frac{3}{4}
\]
\[
\text{Intensity (3)} = \frac{3}{(3)^2} = \frac{3}{9} = \frac{1}{3}
\]

The \textbf{difference} in the two values will give the \textbf{change} in the intensity. We move \textbf{from} two meters \textbf{to} three meters away.

\[
\text{Change} = \text{Intensity (3)} - \text{Intensity (2)} = \frac{1}{3} - \frac{3}{4}
\]

To find the answer, we will need to write these fractions over a common denominator. The LCM of 3 and 4 is 12, so we need to rewrite each fraction with a denominator of 12:

\[
\frac{1}{3} = \frac{4 \cdot 1}{4 \cdot 3} = \frac{4}{12}
\]
\[
\frac{3}{4} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}
\]

So we can rewrite our equation as \(\frac{4}{12} - \frac{9}{12} = -\frac{5}{12}\). The negative value means that the intensity decreases as we move from 2 to 3 meters away.

When moving the detector from two meters to three meters, the intensity falls by \(\frac{5}{12}\) lumens.

\[
\text{Guided Practice}
\]

\[
\text{Example 1}
\]

Elsa baked a small cake for her family. First her sister ate one quarter and her mom ate one third. How much was left for Elsa?

The whole cake is represented by 1. To solve this problem, we subtract the fraction that each person ate.

\[
1 - \frac{1}{4} - \frac{1}{3}
\]

To complete this problem, we must give the terms common denominators. Since the denominators do not share any factors, we simply multiply them together: \(4 \cdot 3 = 12\).
2.2. Rational Numbers in Applications

\[
1 - \frac{1}{4} - \frac{1}{3} \quad \text{Start with the original expression.}
\]
\[
= \frac{12}{12} - \frac{3}{3} - \frac{4}{4} \quad \text{Give each term a common denominator.}
\]
\[
= \frac{12 - 3 - 4}{12} \quad \text{Simplify.}
\]
\[
= \frac{5}{12}
\]

There is \(\frac{5}{12}\) of the original cake left for Elsa.

Review

Which property of addition does each situation involve?

1. Whichever order your groceries are scanned at the store, the total will be the same.
2. However many shovel-loads it takes to move 1 ton of gravel, the number of rocks moved is the same.
3. If Julia has no money, then Mark and Julia together have just as much money as Mark by himself has.

In 4-7, practice your addition and subtraction skills.

4. \(\frac{7}{15} + \frac{2}{6}\)
5. \(\frac{5}{9} + \frac{27}{2}\)
6. \(\frac{5}{7} - \frac{18}{9}\)
7. \(\frac{5}{3} - \frac{1}{4}\)

8. Ilana buys two identically sized cakes for a party. She cuts the chocolate cake into 24 pieces and the vanilla cake into 20 pieces, and lets the guests serve themselves. Martin takes three pieces of chocolate cake and one of vanilla, and Sheena takes one piece of chocolate and two of vanilla. Which of them gets more cake?

9. Nadia, Peter and Ian are pooling their money to buy a gallon of ice cream. Nadia is the oldest and gets the greatest allowance. She contributes half of the cost. Ian is next oldest and contributes one third of the cost. Peter, the youngest, gets the smallest allowance and contributes one fourth of the cost. They figure that this will be enough money. When they get to the check-out, they realize that they forgot about sales tax and worry there will not be enough money. Amazingly, they have exactly the right amount of money. What fraction of the cost of the ice cream was added as tax?

10. The time taken to commute from San Diego to Los Angeles is given by the equation \(\text{time} = \frac{120}{\text{speed}}\), where time is measured in hours and speed is measured in miles per hour (mph). Calculate the change in time that a rush hour commuter would see when switching from traveling by bus to traveling by train, if the bus averages 40 mph and the train averages 90 mph.

Review (Answers)

To view the Review answers, open this PDF file and look for section 2.4.

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for \((2/3) \div (3/4)\) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that \((2/3) \div (3/4) = 8/9\) because \(3/4\) of \(8/9\) is \(2/3\). (In general, \((a/b) \div (c/d) = ad/bc\).) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?
Chapter Outline

3.1 Whole Number Division
3.2 Decimal Addition
3.3 Combinations of Decimal Money Amounts
3.4 Decimal Subtraction
3.5 Multiplication of Decimals and Whole Numbers
3.6 Estimation to Check Decimal Multiplication
3.7 Division of Decimals by Whole Numbers
3.8 Decimal Quotients Using Zero Placeholders
3.9 Decimal Rounding and Division
3.10 Division of Decimals by Decimals
3.11 Factor Pairs
3.12 Greatest Common Factor Using Factor Trees
3.13 Greatest Common Factor Using Lists
3.14 Divisibility Rules to Find Factors
3.15 Least Common Multiple
3.16 Fraction Comparison with Lowest Common Denominators
3.17 Fraction Ordering with Lowest Common Denominators
3.18 Prime Factorization
3.19 Common Multiples

Compute fluently with multi-digit numbers and find common factors and multiples.

6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.

6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

6.NS.4 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express 36 + 8 as 4 (9 + 2).
Jessica won a bet with her friends and received a lunchbox full of mints as her reward. Jessica counts 286 mints in total. She wants this supply to last until the end of the semester, which is 5 weeks away. If she only has mints on school days, how many can Jessica eat per day in order to make her supply last?

In this concept, you will learn how to divide whole numbers.

**Dividing Whole Numbers**

The opposite operation of multiplication is division. To multiply means to add groups of matching things together, to divide means to split up into matching groups.

Let’s look at an example.

\[ 72 \div 9 = \_\_\_ \]

In this problem, 72 is the **dividend** - it is the number being divided. The **divisor** is the number of parts that the dividend is being split into, in this case, 9. The answer to a division problem is called the **quotient**. One way to complete this problem and find the quotient is to recall multiplication facts and work backwards.

To divide 72 by 9, start be asking "What number multiplied by 9 equals 72?"

\[ 9 \times 8 = 72 \]

If 8 groups of 9 equal 72, then of course 72 can be split into 8 groups of 9.
72 ÷ 9 = 8

The quotient is 8.
Here is another example.

15 ÷ 2 =

This is tricky because 15 is not an even number. This means it won’t divide evenly. When this happens and you are using only whole numbers, there will be a remainder.
Start by asking "What number multiplied by 2 comes closest to 15, without going over?"

2 × 7 = 14

So, 7 groups of 2 comes closest to 15, with 1 left over. That is the remainder. Use "r" to show that there is a remainder.

15 ÷ 2 = 7 r 1

The answer is 7 r1.

When dividing larger numbers, it may be easier to keep things organized with a division box.

\[
\begin{array}{c|c}
8 & 825 \\
\hline
8 & 02 \\
\end{array}
\]

Here there is a one digit divisor, 8, and a three digit dividend, 825. You need to figure out how many times 8 goes into 825. To do this, divide the divisor (8) into each digit of the dividend.

\[
\begin{array}{c|c}
8 & 825 \\
\hline
8 & 02 \\
\end{array}
\]

First, divide 8 into 8. Of course the answer is 1. Put the 1 on top of the division box above the 8.

Next, multiply 1 by 8 and subtract the result from the dividend. Then bring down the next number in the dividend (the 2).
Next, look at the next digit in the dividend. There are no 8’s in 2, so put a 0 in the answer, next to the 1.

\[
\begin{array}{c}
10 \\
8)825 \\
-8 \\
\hline
025 \\
\end{array}
\]

Because 8 wouldn’t divide into 2, bring down the next number, 5, and use the two numbers together: 25

Next, look at the next digit in the dividend. There are three 8’s in 25, with a remainder of 1. Add this into the answer.

\[
\begin{array}{c}
103 \\
\hline
8)825 \\
-8 \\
\hline
025 \\
-24 \\
\hline
1 \\
\end{array}
\]

The answer is 103 r1.

You can check your work by multiplying the answer by the divisor.

\[
\begin{array}{c}
103 \\
\times 8 \\
\hline
824 + r \text{ of } 1 = 825 \\
\end{array}
\]

The answer checks out.

You can apply these same steps to any division problem even if the divisor has two or three digits. Work through each value of the dividend with the value of the divisor. Then check your work by multiplying your answer by the divisor.

**Examples**

**Example 1**

Earlier, you were given a problem about Jessica and her minty mother lode.

Jessica wants to make 286 mints last 5 school weeks.

First, figure out how many days Jessica needs to consider in her calculation, since she is only going to eat mints 5 days per week (on school days).

\[5 \times 5 = 25\]

Next, Jessica needs to divide the large number of items by the small number of days.
286 mints ÷ 25 days

Perhaps it will be easier to use a division box

\[ \begin{array}{c}
25)\overline{286} \\
28 \\
\hline
11
\end{array} \]

First, how many times does 25 go into 2? Put a 0 above the answer line. 28

Next, bring down the 8.

\[ \begin{array}{c}
25)\overline{286} \\
28 \\
\hline
11
\end{array} \]

Next, how many times does 25 go into 28? Put a 1 above the answer line.

\[ \begin{array}{c}
25)\overline{286} \\
28 \\
\hline
11
\end{array} \]

Then, bring down the 6.

\[ \begin{array}{c}
25)\overline{286} \\
28 \\
\hline
11
\end{array} \]

Finally, how many times does 25 go into 36? Put a 1 above the answer line.

\[ \begin{array}{c}
25)\overline{286} \\
28 \\
\hline
11
\end{array} \]

36 - 25 = 11. This is the remainder.

The quotient is 11 remainder 11.

Jessica can have 11 mints per day and she will have 11 left at the end of the semester.

**Example 2**

Let’s look at a problem with a two-digit divisor.

The quotient is 204.

You can check your work by multiplying: 204 \times 12.
3.1. Whole Number Division

\[
\begin{array}{c}
204 \\
\times \quad 12 \\
408 \\
+ \quad 2040 \\
\hline
2448
\end{array}
\]

The answer checks out.

**Example 3**

Find the quotient.

\[4 \overline{)469}\]

\[
\begin{array}{c}
117 \ \underline{r} \ 1 \\
4 \ \overline{)469} \\
- 4 \\
- 06 \\
- 4 \quad \text{First, how many times does 4 go into 4? Put a 1 above the answer line.} \\
- \quad 06 \quad \text{Next, bring down the 6.} \\
2 \quad \text{How many times does 4 go into 6? Put a 1 above the answer line.} \\
29 \quad 4 - 6 = 2 \\
- 28 \quad \text{Then bring down the 9. How many times does 4 go into 29? Put a 7 above the answer line.} \\
1 \quad \text{The one left over is the remainder.}
\end{array}
\]

The answer is 117 remainder 1.

**Example 4**

Find the quotient.

\[18 \overline{)3678}\]

108
0204\div 2

18) 3678
First, how many times does 18 go into 3? Put a 0 above the answer line.
36
Next, bring down the 6.
– 36
Next, how many times does 18 go into 36? Put a 2 above the answer line.
07
Then, bring down the 7.
7
Then, how many times does 18 go into 7? Put a 0 above the answer line.
78
Then, bring down the 8.
– 76
Finally, how many times does 18 go into 78? Put a 4 above the answer line.
2 78 – 76 = 2. This is the remainder.

The answer is 204 remainder 2.

Example 5

Find the quotient.
20\cancel{\div} 5020

0251

20) 5020
First, how many times does 20 go into 5? Put a 0 above the answer line.
50
Next, bring down the 0.
– 40
Next, how many times does 20 go into 50? Put a 2 above the answer line.
102
50 – 40 = 10 Bring down the 2.
– 100
Next, how many times does 20 go into 102? Put a 5 above the answer line.
20
100 – 102 = 2. Then, bring down the 0.
– 20
20 goes into 20 once.
0 There is no remainder.

The answer is 251.

Review

Find the quotient.

1. 12 \div 6 = ____
2. 13 \div 4 = ____
3. 132 \div 7 = ____
4. 124 \div 4 = ____
5. 130 \div 5 = ____
6. 216 \div 6 = ____
3.1. Whole Number Division

7. \(1,161 \div 43 = \)____  
8. \(400 \div 16 = \)____  
9. \(1,827 \div 21 = \)____  
10. \(1,244 \div 40 = \)____  
11. \(248 \div 18 = \)____  
12. \(3,264 \div 16 = \)____  
13. \(4,440 \div 20 = \)____  
14. \(7,380 \div 123 = \)____  
15. \(102,000 \div 200 = \)____  
16. \(10,976 \div 98 = \)____

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.4.

Resources

To see the Review answers, open this PDF file and look for section 1.4.
In this concept, you will learn how to add decimals.

Yoshi and his family have decided to have a yard sale. Yoshi’s mother has him collect all the money from the customers. Yoshi wants to make sure that he is collecting the right amount from each person. The first person buys one item for $12.25, a second item for $0.50 and a third item for $0.75. Yoshi knows that he needs to add the decimals but isn’t sure how to begin. How much money does the first person owe Yoshi?

In this concept, you will learn how to add decimals.

**Adding Decimals**

To add decimals, work with the wholes and parts of the numbers separately. Add the parts then add the wholes. The best way to do this is to keep the parts together and keep the wholes together. To do this, simply line up the decimal points in each number that you are adding.

Let’s look at an example.

Add 3.45 + 2.37 = _____

In this problem you have parts and wholes. Let’s rewrite the problem vertically, lining up the decimal points.

\[
\begin{array}{c}
3.45 \\
+ 2.37 \\
\hline
5.82
\end{array}
\]

Next, add the columns vertically and bring the decimal point down into the answer of the problem.

The answer is 5.82.
Let’s look at another example.

\[5 + 3.45 + 0.56 = \, \text{_____}\]

In a problem like this, line up the decimal points, but add zeros to help hold places where there aren’t numbers. This helps keep the addition straight.

First, line up the problem vertically.

\[
\begin{array}{r}
5.00 \\
3.45 \\
+ 0.56 \\
\hline
9.01
\end{array}
\]

Notice that zeros help hold places where you did not have numbers. Now each number in the problem has the same number of digits.

Add the numbers.

The answer is 9.01.

**Examples**

**Example 1**

Earlier, you were given a problem about Yoshi and the yard sale.

Yoshi’s first customer wants to buy 3 different things. To ask for the correct total amount, Yoshi needs to add $12.25, $0.50 and $0.75. Can you find the total amount due?

First, line up the numbers vertically.

\[
\begin{array}{r}
12.25 \\
0.50 \\
+ 0.75 \\
\hline
13.50
\end{array}
\]

Then, add the numbers.

The answer is $13.50. Yoshi must collect $13.50 from the customer.
Example 2

Add the following numbers.
13.25 + 0.80 + 1.30 = _____
First, line up the numbers vertically.

```
  13.25
   0.80
+ 1.30
```

Then, add the numbers.

```
       15.35
```

The answer is 15.35.

Example 3

Add the following decimals.
4.56 + 0.89 + 2.31 = _____
First, line up the numbers vertically.

```
   4.56
   0.89
+ 2.31
```

Then, add the numbers.

```
       7.76
```

The answer is 7.76.

Example 4

Add the following decimals.
5.67 + 0.65 + 0.93 = _____
First, line up the numbers vertically.

```
  5.67
   0.65
+ 0.93
```

```
       7.25
```

The answer is 7.25.
3.2. Decimal Addition

Then, add the numbers.

\[ 7.25 \]

The answer is 7.25.

**Example 5**

Add the following decimals.

\[ 88.92 + 0.57 + 3.12 = _____ \]

First, line up the numbers vertically.

\[
\begin{array}{c}
88.92 \\
0.57 \\
+3.12 \\
\end{array}
\]

Then, add the numbers.

\[ 92.61 \]

The answer is 92.61.

**Review**

Add the following decimals.

1. \( 4.5 + 6.7 = _____ \)
2. \( 3.45 + 2.1 = _____ \)
3. \( 6.78 + 2.11 = _____ \)
4. \( 5.56 + 3.02 = _____ \)
5. \( 7.08 + 11.9 = _____ \)
6. \( 1.24 + 6.5 = _____ \)
7. \( 3.45 + .56 = _____ \)
8. \( 87.6 + 98.76 = _____ \)
9. \( 76.43 + 12.34 = _____ \)
10. \( 5 + 17.21 = _____ \)
11. \( 78 + 13.456 = _____ \)
12. \( .456 + .23 + .97 = _____ \)
13. \( 1.234 + 4.5 + 6.007 = _____ \)
14. \( 3.045 + 3.3 + 9 = _____ \)
15. \( 23 + 4.56 + .0091 = _____ \)

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 3.17.
Jose has four dollars and ninety-four cents. He wants to impress his math teacher by writing the amount of money as a decimal but isn’t sure how to convert from dollars and cents to decimal form. Can you write the amount of money Jose has as a decimal?

In this concept, you will learn how to use decimals to accurately write dollars and cents.

### Writing Dollars and Cents as Decimals

Money is a way that you use decimals every day.

Let’s think about change. Coins are cents. If you have 50 pennies, then you have 50 cents. It takes 100 pennies to make one dollar or one whole. Coins are parts of one dollar so you can represent coins in decimals.

A penny is one cent or it is one out of 100. When you have a collection of pennies, you have so many cents out of 100.

How can you write 5 cents as a decimal? To do this, you need to think about 5 out of 100. You can say that 5 cents is 5 hundredths of a dollar since there are 100 pennies in one dollar.

Let’s write 5 cents as a decimal using place value.

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
<th>Ten Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>.</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The five is in the hundredths box because five cents is five one hundredths of a dollar. You need to add a zero in the tenths box to fill the gap.
### Table 3.2:

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Now you have converted 5 cents to a decimal.

Suppose you have twelve dollars and fourteen cents. A dollar is a whole number amount. Dollars are found to the left of the decimal point. Cents are parts of a dollar. They are found to the right of the decimal point.

There is one ten and the two ones gives you twelve dollars. Then you have some change. One dime and four pennies is equal to fourteen cents.

Here are the numbers:

- 12 wholes
- 14 parts

Let’s put the values in a place value chart.

### Table 3.3:

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>.</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

There is the money amount. The answer is $12.14. Notice that the dollar sign is added to the answer to represent money.

### Example 1

Earlier, you were given a problem about Jose and his four dollars and ninety-four cents.

How can you write that value as a decimal?

First, write the amount in a place value chart.

### Table 3.4:

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>.</td>
<td>9</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then, write the number as a decimal.

4.94

The answer is $4.94.

### Example 2

Write 75 cents as a decimal.
First, think about what part of a dollar 75 cents is. Seventy-five cents is seventy-five out of 100. Now, you can put this into the place value chart.

**Table 3.5:**

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
<th>Ten Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>.</td>
<td>7</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Then write the number as a decimal.

0.75

The answer is $0.75.

**Example 3**

Write three dollars and fifteen cents as a decimal.

First, write the amount in a place value chart.

**Table 3.6:**

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
<th>Ten Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>.</td>
<td>1</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Then, write the number as a decimal.

3.15

The answer is $3.15

**Example 4**

Write eighty-nine cents as a decimal.

First, write the amount in a place value chart.

**Table 3.7:**

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
<th>Ten Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>.</td>
<td>8</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

Then, write the number as a decimal.

0.89

The answer is $0.89.
Example 5

Write fifteen dollars and twenty-five cents as a decimal.
First, write the amount in a place value chart.

<table>
<thead>
<tr>
<th>Hundred</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
<th>Ten Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>.</td>
<td>2</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then, write the number as a decimal.
15.25
The answer is $15.25.

Review

Write the following money amounts as decimals.

1. Ten dollars and fifty cents
2. Five dollar bills, two quarters and three pennies
3. A twenty dollar bill, a ten dollar bill and three dimes
4. Fifteen nickels and two quarters
5. Six dollar bills, three quarters and a nickel
6. A dime and fifteen pennies
7. Two dollars, ten dimes and two quarters
8. Three quarters, five nickels and ten pennies
9. Sixteen dollar bills, two quarters, a nickel and ten pennies
10. A ten dollar bill, a five dollar bill, one quarter, three nickels, a dime and ten pennies
11. A ten dollar bill, a quarter, a dime, five nickels and fifteen pennies
12. A five dollar bill, two quarters, two dimes, a nickel and two pennies
13. Two dollar bills, a quarter and five pennies
14. Three dollar bills, a twenty dollar bill, two quarters and five dimes
15. A dollar, three quarters, two dimes, five nickels and two pennies

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.4.

Resources

Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/167356
3.4 Decimal Subtraction

In this concept, you will learn how to subtract decimals.

Ivana goes to the thrift store with $20 in her pocket. She purchases a shirt and a necklace that total $12.42. Ivana hands the cashier her $20 bill and watches as the cashier enters the number. Ivana tries to predict the amount of change that she will receive from the cashier, but she can’t decide what to do since $12.42 and $20 have a different number of digits. Can you find the difference between $20 and $12.42 and figure out much change the cashier should give Ivana?

In this concept, you will learn how to subtract decimals.

**Subtracting Decimals**

To subtract decimals, you will work with the wholes and parts of the numbers separately. First, you subtract the parts and then you subtract the wholes. The best way to do this is to keep the parts together and keep the wholes together. To do this, simply line up the decimal points in each number that you are subtracting.

Let’s look at an example.

\[ 6.78 - 2.31 = \_\_\_\_ \]

First, line up the problem vertically.

\[
\begin{array}{c}
6.78 \\
- 2.31 \\
\hline
\end{array}
\]

Then, subtract each digit vertically.

\[
\begin{array}{c}
6.78 \\
- 2.31 \\
\hline
4.47 \\
\end{array}
\]
The answer is 4.47.

Sometimes, the values in a subtraction problem can have a different number of digits. Add zeros to help hold places where there are no digits. That way each number has the same number of places.

Let’s look at an example.

67.89 - 18.4 = _____

First, line up the problem vertically with the decimal point.

\[
\begin{array}{c}
67.89 \\
18.40 \\
\hline
49.49
\end{array}
\]

Then, subtract each digit vertically.

49.49

The answer is 49.49.

**Examples**

**Example 1**

Earlier, you were given a problem about Ivana and her purchases at the thrift store.

She has $20 and she spends $12.42. How can she subtract her total and calculate her change?

\[
\begin{array}{c}
$20 \\
- 12.42 \\
\hline
7.58
\end{array}
\]

First, add zeros to the 20 so that both numbers have the same number of places.

20.00

Next, line up the numbers vertically.

\[
\begin{array}{c}
20.00 \\
\hline
-12.42
\end{array}
\]

Then, subtract the numbers

7.58

The answer is $7.58. Ivana should receive $7.58 from the cashier.
Example 2

Subtract the decimals.
12.5 - 10.38 = _____

First, add a zero to the 12.5 so that both numbers have the same number of places.

\[
\begin{array}{c}
12.5 \\
\end{array}
\]

Next, line up the numbers vertically.

\[
\begin{array}{c}
12.50 \\
-10.38 \\
\end{array}
\]

Then, subtract the numbers.

\[
\begin{array}{c}
2.12 \\
\end{array}
\]

The answer is 2.12.

Example 3

Subtract the decimals.
16 - 12.22 = _____

First, add zeros to the 16 so that both numbers have the same number of places.

\[
\begin{array}{c}
16.00 \\
\end{array}
\]

Next, line up the numbers vertically.

\[
\begin{array}{c}
16.00 \\
-12.22 \\
\end{array}
\]

Then, subtract the numbers.

\[
\begin{array}{c}
3.78 \\
\end{array}
\]

The answer is 3.78.
Example 4

Subtract the decimals.
18.86 - 13.45 = _____
First, line up the numbers vertically.

\[
\begin{array}{c}
18.86 \\
-13.45 \\
\hline
\end{array}
\]

Then, subtract the numbers.

\[5.41\]

The answer is 5.41

Example 5

Subtract the decimals.
19.2 - 13.211 = _____
First, add zeros to the 19.2 so that both numbers have the same number of places.

\[
\begin{array}{c}
19.200 \\
-13.211 \\
\hline
\end{array}
\]

Next, line up the numbers vertically.

\[5.989\]

The answer is 5.989.

Review

Subtract the following decimals.

1. 17.65 - 4 = _____
2. 18.97 - 3.4 = _____
3. 22.50 - .78 = _____
4. 27.99 - 1.99 = _____
5. $33.11 - 3.4 = \_\_\_\_\_\_\_\_
6. $44.59 - 11.34 = \_\_\_\_\_\_\_
7. $78.89 - 5 = \_\_\_\_\_\_\_
8. $222.56 - 11.2 = \_\_\_\_\_\_\_
9. $567.09 - 23.4 = \_\_\_\_\_\_\_
10. $657.80 - 3.04 = \_\_\_\_\_\_\_
11. $345.01 - 123.90 = \_\_\_\_\_\_\_
12. $567.08 - 111.89 = \_\_\_\_\_\_\_
13. $378.99 - 345.12 = \_\_\_\_\_\_\_
14. $786.01 - 123.10 = \_\_\_\_\_\_\_
15. $504.32 - 345.89 = \_\_\_\_\_\_\_

**Review (Answers)**

To see the Review answers, open this [PDF file](#) and look for section 3.18.

**Resources**

[18.01 - 2.953](#)

*Mathmeeting.com*

**MEDIA**

Click image to the left or use the URL below.

URL: [https://www.ck12.org/flx/render/embeddedobject/167931](https://www.ck12.org/flx/render/embeddedobject/167931)
3.5 Multiplication of Decimals and Whole Numbers

In this concept, you will learn how to multiply decimals and whole numbers together.

Mrs. Andersen is planning a field trip to the Science Museum for her sixth grade class. The group rate for a student ticket is $8.95 per ticket. If 22 students attend the field trip, how much money should Mrs. Andersen have to cover the cost of admission for the students?

In this concept, you will learn how to multiply decimals and whole numbers together.

Multiplying Decimals and Whole Numbers

Multiplication is a short-cut for repeated addition. It is a way to add the same number of groups several or “multiple” times.

Here is a multiplication equation.

\[ 4 \times 3 = 12 \]

This equation tells you that there are 3 groups of 4 or 4 groups of 3. Either grouping results in the same total. The product of 4 multiplied by 3 is 12. A product is the result of multiplying two or more numbers. You can also think of multiplying whole numbers and decimals as groups.

Here is a whole number and decimal multiplication problem.

\[ 2(0.25) = \text{_____} \]

Remember that the parentheses tell you the order of operation and can also indicate multiplication. Think of 2 times 0.25 as two groups of twenty-five hundredths. Let’s look at this on a hundreds grid.
Each color represents a group of twenty-five hundredths. Two groups of twenty-five hundredths total fifty hundredths.

\[2(0.25) = 0.50\]

This is one way to multiply decimals and whole numbers. You can also multiply a decimal and a whole number just like you would two whole numbers. Afterwards, add the decimal point in the product the same number of places in the decimal number multiplier.

Here is another whole number and decimal multiplication problem.

\[4(1.25) = \underline{_____}\]

First, ignore the decimal and multiply as if this were two whole numbers.

\[
\begin{array}{c}
1.25 \\
\times \ 4 \\
\hline
500 \\
\end{array}
\]

Then, add the decimal point in the product. There are two decimal places in 1.25. Count two places from right to left and place the decimal point in the product.
The product of 4 times 1.25 is 5.00.

Examples

Example 1

Earlier, you were given a problem about Mrs. Andersen’s field trip to the museum. Mrs. Andersen needs to find the total cost of 22 tickets that cost $8.95 each. First, write a multiplication problem to find the total cost.

\[
22 \times 8.95 = \\
\]

Then, ignore the decimal and multiply as if this were two whole numbers.

\[
\begin{array}{c}
8.95 \\
\times 22 \\
\hline
1790 \\
+1790 \\
\hline
19690 \\
\end{array}
\]

Next, add the decimal point in the product. There are two decimal places in 8.95.

\[
\begin{array}{c}
8.95 \\
\times 22 \\
\hline
1790 \\
+1790 \\
\hline
196.90 \\
\end{array}
\]

The total cost for 22 student tickets at the group rate is $196.90.

Example 2

Find the product for the following problem.

9 friends decided to go to a movie on Friday night. They each paid the $8.50 for admission. How much money did they spend in all? First, write a multiplication problem to find the total cost.
9(8.50) = _____

Then, ignore the decimal and multiply as if this were two whole numbers.

\[
\begin{array}{c}
\phantom{6.50} \\
\times 9 \\
\end{array}
\]

Next, add the decimal point in the product. There are two decimal places in 8.50.

\[
\begin{array}{c}
8.50 \\
\times 9 \\
\end{array}
\]

The cost of 9 movie tickets was $76.50.

**Example 3**

Find the product.

\[
3(4.52) = _____
\]

First, ignore the decimal and multiply as if this were two whole numbers.

\[
\begin{array}{c}
\phantom{4.52} \\
\times 3 \\
\end{array}
\]

Then, add the decimal point in the product.

\[
\begin{array}{c}
4.52 \\
\times 3 \\
\end{array}
\]

The product of 3 times 4.52 is 13.56.

**Example 4**

Find the product.
5(2.34) = ____

First, ignore the decimal and multiply as if this were two whole numbers.

\[
\begin{array}{c}
2.34 \\
\times 5 \\
\hline
1170 \\
\end{array}
\]

Then, add the decimal point in the product.

\[
\begin{array}{c}
2.34 \\
\times 5 \\
\hline
11.70 \\
\end{array}
\]

The product of 5 times 2.34 is 11.70.

**Example 5**

Find the product.

7(3.561) = ____

First, ignore the decimal and multiply as if this were two whole numbers.

\[
\begin{array}{c}
3.561 \\
\times 7 \\
\hline
24927 \\
\end{array}
\]

Then, add the decimal point in the product.

\[
\begin{array}{c}
3.561 \\
\times 7 \\
\hline
24.927 \\
\end{array}
\]

The product of 7 times 3.561 is 24.927.

**Review**

Find the product for the following problems.
3.5. Multiplication of Decimals and Whole Numbers

1. \(5(1.24) = \) 
2. \(6(7.81) = \) 
3. \(7(9.3) = \) 
4. \(8(1.45) = \) 
5. \(9(12.34) = \) 
6. \(2(3.56) = \) 
7. \(6(7.12) = \) 
8. \(3(4.2) = \) 
9. \(5(2.4) = \) 
10. \(6(3.521) = \) 
11. \(2(3.222) = \) 
12. \(3(4.223) = \) 
13. \(4(12.34) = \) 
14. \(5(12.45) = \) 
15. \(3(143.12) = \) 
16. \(4(13.672) = \) 
17. \(2(19.901) = \) 
18. \(3(67.321) = \)

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.1.

Resources

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/169496

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/169497
Here you’ll learn how to use estimation to check the reasonableness of an answer when you multiply decimals.

When Dylan’s baby sister was born last month she weighed 6 pounds 8.7 ounces, or 6.54375 pounds. Dylan’s mom said that his sister will probably triple her weight by the time she is a year old. How could Dylan estimate what his sister will weigh when she is one year old?

In this concept, you will learn to estimate decimal products by multiplying leading digits.
Estimating to Check Decimal Multiplication

Recall that when you estimate you are finding an approximate solution to a problem. One good time to use estimation is when you want to check your answer to a problem. If you estimate a solution to a problem either before or after you solve it, you can see whether or not your answer is realistic.

When you are working with decimal numbers with many digits, one way to estimate their product is to multiply only their two leading digits.

Here are the steps for multiplying decimal numbers using leading digits.

1. Identify the two leading (left-most) digits of each number, preserving the location of the decimal point. Note that if 0 is the only digit to the left of a decimal point, it does not count as one of the leading digits.
2. Multiply the leading digits using your steps for decimal multiplication.

Here is an example.

Estimate the product of \( 6.42 \times 0.383 \).

First, identify the two leading digits of each number.

- The leading digits of 6.42 are 6.4
- The leading digits of 0.383 are 0.38

Note that because 0 is the only digit to the left of the decimal point in the second number, it does not count as one of the leading digits.

Now, multiply just as if you were multiplying whole numbers. Because the original numbers have 3 digits total after their decimal points, insert a decimal point into your answer so that it has 3 digits to its right.

\[
\begin{array}{c}
6.4 \\
\times \\
0.38 \\
\hline
512 \\
1920 \\
\hline
2.432
\end{array}
\]

The answer is the product of 6.42 and 0.383 is approximately 2.432.

Here is another example.

Find the product. Then estimate to confirm your solution. \( 22.17 \times 4.45 \).

First, notice that this problem asks you to do two things. You will need to multiply and then estimate the solution. Start by multiplying just as if you were multiplying whole numbers. Because the original numbers have 4 digits total after their decimal points, insert a decimal point into your answer so that it has 4 digits to its right.

\[
\begin{array}{c}
22.17 \\
\times \\
4.45 \\
\hline
11085 \\
88680 \\
\hline
98.6565
\end{array}
\]

Next, estimate the answer by multiplying only the leading digits. Identify the two leading digits of each number.
• The leading digits of 22.17 are 22
• The leading digits of 4.45 are 4.4

Now, multiply the leading digits.

\[
\begin{array}{c}
22 \\
\times 4.4 \\
\hline
88 \\
+ 880 \\
\hline
96.8
\end{array}
\]

Next, compare your answer with your estimate. Your answer of 98.6565 is close to your estimate of 96.8. This means you can be confident that your answer is correct.

The answer is that the exact product of 22.17 \times 4.45 is 98.6565, while the estimate of the product is 96.8.

**Examples**

**Example 1**

Earlier, you were given a problem about Dylan and his baby sister. When she was born she weighed 6.54375 pounds and she will probably triple her weight by the time she is a year old. Dylan wants to estimate how much she will weigh when she is a year old.

First, Dylan needs to realize that if his sister triples her weight that means her original weight will be multiplied by 3. Dylan is trying to find the product of 6.54375 \times 3.

Now, Dylan can estimate the product by multiplying the leading digits. He should identify the two leading digits of each number.

• The leading digits of 6.54375 are 6.5
• The leading digits of 3 are 3

Note that because 3 only had one digit to start with, it only has one leading digit!

Next, Dylan should multiply just as if he was multiplying whole numbers. Because the original numbers have 1 digit total after their decimal points, he should insert a decimal point into his answer so that it has 1 digit to its right.

\[
\begin{array}{c}
6.5 \\
\times 3 \\
\hline
19.5
\end{array}
\]

The answer is that Dylan’s baby sister will likely weigh approximately 19.5 pounds when she is a year old.

**Example 2**

Estimate the product of 0.4561 \times 0.32109.

First, identify the two leading digits of each number. Remember that when the only digit to the left of the decimal point is a 0, it does not count as a leading digit.

• The leading digits of 0.4561 are 0.45
• The leading digits of 0.32109 are 0.32

Now, multiply just as if you were multiplying whole numbers. Because the original numbers have 4 digits total after their decimal points, insert a decimal point into your answer so that it has 4 digits to its right.

\[
\begin{array}{c}
.45 \\
\times .32 \\
\hline
90 \\
+ 1350 \\
\hline
.1440
\end{array}
\]

Your estimate is .1440 which is the same as 0.144.

The answer is that the product of 0.4561 and 0.32109 is approximately 0.1440.

**Example 3**

Find the product. Then estimate to confirm your solution. \(6.79 \times 1.2\).

First, multiply to find the exact answer. You will multiply just as if you were multiplying whole numbers. Because the original numbers have 2 digits total after their decimal points, insert a decimal point into your answer so that it has 2 digits to its right.

\[
\begin{array}{c}
67.9 \\
\times 1.2 \\
\hline
1358 \\
+ 6790 \\
\hline
81.48
\end{array}
\]

Next, estimate the answer by multiplying the leading digits. Identify the two leading digits of each number.

• The leading digits of 67.9 are 67
• The leading digits of 1.2 are 1.2

Now, multiply the leading digits.

\[
\begin{array}{c}
67 \\
\times 1.2 \\
\hline
134 \\
+ 670 \\
\hline
80.4
\end{array}
\]

Next, compare your answer with your estimate. Your answer of 81.48 is close to your estimate of 80.4. This means you can be confident that your answer is correct.

The answer is that the exact product of 67.9 \(\times\) 1.2 is 81.48, while the estimate of the product is 80.4.

**Example 4**

Estimate the product of 5.321 \(\times\) 2.301.

First, identify the two leading digits of each number.
• The leading digits of 5.321 are 5.3
• The leading digits of 2.301 are 2.3

Now, multiply just as if you were multiplying whole numbers. Because the original numbers have 2 digits total after their decimal points, insert a decimal point into your answer so that it has 2 digits to its right.

\[
\begin{array}{c}
5.3 \\
\times 2.3 \\
\hline
159 \\
+ 1060 \\
\hline 12.19
\end{array}
\]

The answer is that the product of 5.321 and 2.301 is approximately 12.19.

**Example 5**

Find the product. Then estimate to confirm your solution. \(9.1204 \times 8.713\).

First, multiply to find the exact answer. You will multiply just as if you were multiplying whole numbers. Because the original numbers have 7 digits total after their decimal points, insert a decimal point into your answer so that it has 7 digits to its right.

\[
\begin{array}{c}
9.1204 \\
\times 8.713 \\
\hline 273612 \\
912040 \\
63842800 \\
+ 729632000 \\
\hline 79.4660452
\end{array}
\]

Next, estimate the answer by multiplying the leading digits. Identify the two leading digits of each number.

• The leading digits of 9.1204 are 9.1
• The leading digits of 8.713 are 8.7

Now, multiply the leading digits.

\[
\begin{array}{c}
9.1 \\
\times 8.7 \\
\hline 637 \\
+ 7280 \\
\hline 79.17
\end{array}
\]

Next, compare your answer with your estimate. Your answer of 79.4660452 is close to your estimate of 79.17. This means you can be confident that your answer is correct.

The answer is that the exact product of 9.1204 \(\times\) 8.713 is 79.4660452, while the estimate of the product is 79.17.
3.6. *Estimation to Check Decimal Multiplication*

**Review**

Estimate the products by multiplying the leading digits.

1. $7.502 \times 0.9281$
2. $46.14 \times 2.726$
3. $0.39828 \times 0.16701$
4. $83.243 \times 6.517$
5. $5.67 \times .987$
6. $7.342 \times 1.325$
7. $17.342 \times .325$
8. $.34291 \times 1.525$
9. $.5342 \times .87325$
10. $.38942 \times .9825$
11. $7.567 \times 3.325$
12. $12.342 \times 11.325$
13. $21.342 \times 14.555$
14. $.110342 \times .098325$
15. $37.1342 \times 1.97325$

**Answers for Review Problems**

To see the Review answers, open this [PDF file](#) and look for section 2.11.
In this concept, you will learn how to divide decimals by whole numbers.

Jack has 15.62 yards of fabric leftover from a project. He wants to make blankets and donate them to the animal shelter. He thinks he can make 11 blankets. How long will each blanket be?

In this concept, you will learn how to divide decimals by whole numbers.

**Dividing Decimals by Whole Numbers**

To **divide** means to split up into equal parts. Dividing a decimal number means splitting up the decimal value into sections.

Here is a division problem.

\[ 4.64 \div 2 = \_\_\_\_ \]

In this problem, 4.64 is the **dividend**. The dividend is the number being divided. The **divisor** is the number of parts the dividend is being divided into. The divisor is 2. Remember that the divisor goes outside of the division box in long division.

Dividing a decimal is similar to whole number division. First, divide the 2 into each number. Ignore the decimal for now.
3.7. Division of Decimals by Whole Numbers

Then, insert the decimal point into the quotient. Bring the decimal point from its place in the division box directly up into the quotient.

\[
\begin{array}{c}
2.32 \\
2longdiv4.64 \\
\end{array}
\]

The quotient of 4.64 divided by 2 is 2.32.

**Examples**

**Example 1**

Earlier, you were given a problem about Jack and blankets.

Jack wants to make 11 blankets from 15.62 yards of fabric. Divide 15.62 by 11 to find the length of each blanket. First, ignore the decimal and divide as if this were two whole numbers.

\[
\begin{array}{c}
142 \\
11longdiv15.62 \\
\end{array}
\]

Then, add decimal point into the quotient. Bring the decimal point from its place in the division box up into the quotient.

\[
\begin{array}{c}
1.42 \\
11longdiv15.62 \\
\end{array}
\]

Each blanket will be 1.42 yards long.

**Example 2**

Find the quotient.
$66.3 \div 3$

First, ignore the decimal and divide as if this were two whole numbers.

\[
\begin{array}{c}
221 \\
3longdiv66.3 \\
6 \\
06 \\
6 \\
03 \\
-3 \\
0
\end{array}
\]

Then, add decimal point into the quotient. Bring the decimal point from its place in the division box up into the quotient.

\[
\begin{array}{c}
22.1 \\
3longdiv66.3 \\
\end{array}
\]

Find the quotient for the following problems.

**Example 3**

$36.48 \div 12$

First, ignore the decimal and divide as if this were two whole numbers.

\[
\begin{array}{c}
304 \\
12longdiv36.48 \\
36 \\
04 \\
-0 \\
48 \\
-48 \\
0
\end{array}
\]

Then, add decimal point into the quotient. Bring the decimal point from its place in the division box up into the quotient.

\[
\begin{array}{c}
3.04 \\
12longdiv36.48 \\
\end{array}
\]

The quotient of $36.48$ divided by $12$ is $3.04$. 
Example 4

\[ 2.46 \div 3 \]

First, ignore the decimal and divide as if this were two whole numbers.

\[
\begin{array}{c|c}
& 2 & 3 \\
\hline
5 & 1 & 5 \\
\hline
-1 & 0 & 15 \\
\hline
1 & 5 & 0 \\
\end{array}
\]

Then, add decimal point into the quotient. Bring the decimal point from its place in the division box up into the quotient.

\[
\begin{array}{c|c}
& 2 & 3 \\
\hline
5 & 1 & 5 \\
\end{array}
\]

The quotient of 2.46 divided by 3 is 0.82.

Example 5

\[ 11.5 \div 5 \]

First, ignore the decimal and divide as if this were two whole numbers.

\[
\begin{array}{c|c}
& 2 & 3 \\
\hline
5 & 1 & 5 \\
\hline
-1 & 0 & 15 \\
\hline
1 & 5 & 0 \\
\end{array}
\]

Then, add decimal point into the quotient. Bring the decimal point from its place in the division box up into the quotient.

\[
\begin{array}{c|c}
& 2 & 3 \\
\hline
5 & 1 & 5 \\
\end{array}
\]

The quotient of 11.5 divided by 5 is 2.3.
Review

Find the quotient for the following problems.

1. $36.48 ÷ 2$
2. $5.4 ÷ 3$
3. $14.16 ÷ 6$
4. $18.63 ÷ 3$
5. $11.6 ÷ 4$
6. $11.26 ÷ 2$
7. $27.6 ÷ 4$
8. $18.5 ÷ 5$
9. $49.2 ÷ 4$
10. $27.09 ÷ 7$
11. $114.4 ÷ 8$
12. $325.8 ÷ 9$
13. $107.6 ÷ 4$
14. $115.7 ÷ 5$
15. $192.6 ÷ 6$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.10.

Resources

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/166237
3.8 Decimal Quotients Using Zero Placeholders

In this concept, you will learn how to divide a decimal using zero place holders.

Jerry is hosting a barbecue for his friends. He buys 10.5 pounds of ground meat to make hamburgers. He wants to make 20 hamburger patties for the party. How much meat will be in each patty if he divides the meat evenly?

In this concept, you will learn how to divide a decimal using zero place holders.

Zero Placeholder

Not all decimal division problems will divide exactly. In some cases, a division problem will have a remainder. A remainder is the amount left over when the dividend is not equally divided by the divisor. The remainder is written as “R” followed by a number.

Here is a division problem with a remainder.

\[
\begin{array}{c}
2.9R4 \\
5)14.9 \\
-10 \\
4.9 \\
-4.5 \\
4 \\
\end{array}
\]

The quotient of 14.9 divided by 5 is 2.9 with a remainder of 4. You can continue dividing the dividend to find a more accurate quotient. Place a zero as a placeholder at the end of the decimal number in the division box.

Here is the same division problem with a zero placeholder.
The quotient of 14.9 divided by 5 is 2.98. This means that 5 can go into 14.9 exactly 2.98 times.
Adding zero placeholders at the end of a decimal does not change the value of the decimal.

\[ 14.9 = 14.9000 \]

Zero placeholders can be used to continue dividing to get a more accurate quotient. You can also add as many zero placeholders as you need.

**Examples**

**Example 1**

Earlier, you were given a problem about Jerry and the hamburgers.
Jerry wants to make 20 hamburger patties from 10.5 pounds of ground meat. Divide 10.5 by 20 to find the exact weight of each patty.
First, start dividing 10.5 by 20.

\[
\begin{align*}
5 & \overline{)14.90} \\
\frac{-10}{4.9} \\
\frac{-4.5}{0.40} \\
\frac{-4.0}{0}
\end{align*}
\]

Next, add the decimal point to the quotient.

\[ 0.525 \]

Each patty will weigh exactly 0.525 pounds.
3.8. Decimal Quotients Using Zero Placeholders

**Example 2**

Divide and use zero placeholders if needed.

\[ 3.1 \div 8 = \_\_\_\_\_ \]

First, start dividing 3.1 by 8. Remember that 3.1 is the dividend and 8 is the divisor.

\[
\begin{array}{c}
4 \\ 8 \) 3.1 \\
-28 \\
3 \\
\end{array}
\]

Then, add zero placeholders and continue to divide.

\[
\begin{array}{c}
3875 \\
8 \) 3.1000 \\
-24 \\
70 \\
-64 \\
60 \\
-56 \\
40 \\
-40 \\
0 \\
\end{array}
\]

Then, add the decimal point to the quotient.

\[
\begin{array}{c}
0.3875 \\
8 \) 3.1000 \\
\end{array}
\]

The final answer for 3.1 divided by 8 is 0.3875.

**Divide the following problems and use zero placeholders if needed.**

**Example 3**

\[ 13.95 \div 6 = \_\_\_\_\_ \]

First, start dividing 13.95 by 6.

\[
\begin{array}{c}
232 \\
6 \) 13.95 \\
-12 \\
19 \\
-18 \\
15 \\
-12 \\
3 \\
\end{array}
\]
Then, add zero placeholders and continue to divide.

\[
\begin{array}{c|c|c}
& 2 & 3 & 2 & 5 \\
\hline
6 & 1 & 3 & .9 & 5 & 0 \\
\hline
& - & 1 & 2 \\
\hline
& & 1 & 9 \\
\hline
& & - & 1 & 8 \\
\hline
& & & 1 & 5 \\
\hline
& & & - & 1 & 2 \\
\hline
& & & & 3 & 0 \\
\hline
& & & & - & 3 & 0 \\
\hline
& & & & & 0 \\
\end{array}
\]

Next, add the decimal point to the quotient.

\[
\begin{array}{c|c|c}
& 2 & .3 & 2 & 5 \\
\hline
6 & 1 & 3 & .9 & 5 & 0 \\
\hline
& - & 1 & 2 \\
\hline
& & 1 & 9 \\
\hline
& & - & 1 & 8 \\
\hline
& & & 1 & 5 \\
\hline
& & & - & 1 & 2 \\
\hline
& & & & 3 & 0 \\
\hline
& & & & - & 3 & 0 \\
\hline
& & & & & 0 \\
\end{array}
\]

The quotient of 13.95 divided by 6 is 2.325.

**Example 4**

\[
\begin{array}{c|c|c}
& 2 & .5 \\
\hline
2 & \div & 2 \\
\hline
& - & 2 \\
\hline
& & 0 & 5 \\
\hline
& & - & 4 \\
\hline
& & & 1 \\
\end{array}
\]

First, start dividing 2.5 by 2.

Then, add zero placeholders and continue to divide.

\[
\begin{array}{c|c|c}
& 1 & 2 & 5 \\
\hline
2 & \div & 2 .5 & 0 \\
\hline
& - & 2 \\
\hline
& & 0 & 5 \\
\hline
& & - & 4 \\
\hline
& & & 1 & 0 \\
\hline
& & & - & 1 & 0 \\
\hline
& & & & 0 \\
\end{array}
\]

Next, add the decimal point to the quotient.
The quotient of 2.5 divided by 2 is 1.25.

**Example 5**

\[ 1.66 \div 4 = \Box \]

First, start dividing 1.66 by 4.

\[
\begin{array}{c}
41 \\
4)1.66 \\
-16 \\
\hline
06 \\
-4 \\
\hline
2 \\
\end{array}
\]

Then, add zero placeholders and continue to divide.

\[
\begin{array}{c}
415 \\
4)1.660 \\
-16 \\
\hline
06 \\
-4 \\
\hline
20 \\
-20 \\
\hline
0 \\
\end{array}
\]

Next, add the decimal point to the quotient.

\[ 0.415 \]

The quotient of 1.66 divided by 4 is 0.415.

**Review**

Divide the following problems and use zero placeholders if needed.

1. 5)17.6
2. 4)12.3
3. 4)14.4
4. 5)27.51
5. 6)13.6
6. 6)54.9
7. 8)4.18
8. 8)94.1
9. 8)10.04
10. 4)24.89
11. 12)27.9

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.11.

Resources

MEDIA
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Click image to the left or use the URL below.
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Vincent is looking at a cupcake recipe. The recipe makes a dozen cupcakes, but he only wants to make 4 cupcakes. He knows that 4 out of 12 is also one-third. If he divides the ingredients in the recipe by 3, he will have enough ingredients to make just 4 cupcakes. The recipe calls for 215 grams of sugar and his kitchen scale only measures grams to the tenths place. How much sugar should Vincent measure out to make a third of the recipe?

In this concept, you will learn to divide and round decimals.

**Rounding Decimals When Dividing**

You can divide decimals by whole numbers and use zero placeholders to find the most accurate quotient. Some quotients are so long that it is difficult to decipher the value of the decimal. Rounding the decimal number to a decimal place value would make the number easier to evaluate.

Remember to look at the number to the right of the place value you are rounding to. If the number is less than 5, round the number down. If the number is equal to or greater than 5, round the number up.

Round 2.1046891425 to the nearest hundredths place. The number to the right of the hundredths place is 4. Round the number down.

\[
2.1046891425 \approx 2.10
\]

Round 2.1046891425 to the nearest ten-thousandths place. The number to the right of the ten-thousandths place is 8. Round the number up.

\[
2.1046891425 \approx 2.1047
\]

Here is a decimal division problem. Round the quotient to the nearest ten-thousandths place.
1.265 ÷ 4

First, divide to find the quotient. Remember to bring up the decimal point in the same location as the dividend.

\[
\begin{array}{r}
4 & \overset{0.31625}{\overline{1.26500}} \\
\underline{-12} & \\
06 & \\
\underline{-4} & \\
02 & \\
\underline{-25} & \\
07 & \\
\underline{-24} & \\
10 & \\
\underline{-8} & \\
20 & \\
\underline{-20} & \\
0 & \\
\end{array}
\]

Then, take the quotient and round it to the nearest ten-thousandths place. Look at the digit to the right of the ten-thousandths; it is 5. Round the number up.

\[0.31625 \approx 0.3163\]

The quotient of 1.265 divided by 4 rounded to the nearest ten-thousandths place is 0.3163.

Keep in mind some decimal quotients can be quite long.

Here is a division problem with a very long quotient. Round the quotient to thousandths.

\[13.87 ÷ 7 = 1.9814285714285713\]
\[1.9814285714285713 \approx 1.981\]

The quotient of 13.87 divided by 7, rounded to the thousandths is 1.981

If you know you will round the quotient, you can stop dividing after you’ve found the digit to the right of the place value you are rounding to. Take for example the problem above. Stop dividing once you’ve found the digit in the ten-thousandths place. The digits that come afterwards will not affect the rounding of the decimal number.
Round 1.9814 to the thousandths place.

\[
\begin{align*}
1.9814 & \approx 1.981 \\
\end{align*}
\]

The answer is the same, 1.981.

**Examples**

**Example 1**

Earlier, you were given a problem about Vincent making cupcakes. He is trying to make a third of a cupcake recipe by dividing the ingredients by 3. Since his kitchen scale only measures to a tenths of a gram, find a third of 215 grams of sugar, rounded to the tenths place.

First, divide to find the quotient. You can stop at the hundredths place.

\[
\begin{align*}
3) & 215.00 \\
-21 & \\
\underline{05} & \\
-3 & \\
\underline{20} & \\
-18 & \\
\underline{20} & \\
-18 & \\
\underline{2} & \\
\end{align*}
\]

Then, round the quotient to the tenths place.

\[
\begin{align*}
71.66 & \approx 71.7 \\
\end{align*}
\]
Example 2

Find the quotient and round it to the nearest thousandth.

\[0.45622 \div 4\]

First, divide to find the quotient. Add zero placeholders if necessary. Remember to bring up the decimal point in the same location as the dividend. Divide all the way or you can stop dividing once you’ve found the digit in the ten-thousandths place.

\[
\begin{array}{c|c}
4 & 0.456220 \\
-4 & -4 \hline
05 & 0 \hline
-4 & -0 \hline
16 & 22 \hline
-16 & -20 \hline
02 & 20 \hline
-16 & 20 \hline
02 & 0 \hline
\end{array}
\]

OR

\[
\begin{array}{c|c}
4 & 0.45622 \\
-4 & -4 \hline
05 & 0 \hline
-4 & -0 \hline
16 & 2 \hline
-16 & -2 \hline
02 & 0 \hline
\end{array}
\]

Then, round the quotient to the thousandths place. Look at the digit to the right in the ten-thousandths place; it is 0. Round the number down.

\[0.114055 \approx 0.114 \quad \text{OR} \quad 0.1140 \approx 0.114\]

The quotient of 0.45622 divided by 4 rounded to the thousandths place is 0.114. Both methods will bring you to the same conclusion. The method on the right may be quicker if you only need a rounded quotient.

Find the quotient rounded to the nearest thousandths for the following problems.
Example 3

\[
0.51296 \div 2 = 
\]

First, divide to find the quotient.

\[
\begin{array}{c}
\text{2)0.51296} \\
\underline{-4} \\
\text{11} \\
\underline{-10} \\
\text{12} \\
\underline{-12} \\
\text{09} \\
\underline{-8} \\
\text{16} \\
\underline{-16} \\
\text{0}
\end{array}
\]

Then, round the quotient to the thousandths place.

\[
0.25648 \approx 0.256
\]

The quotient of 0.51296 divided by 2 is 0.256.

Example 4

\[
10.0767 \div 3 = 
\]

First, divide to find the quotient.

\[
\begin{array}{c}
\text{3)10.0767} \\
\underline{-9} \\
\text{10} \\
\underline{-9} \\
\text{17} \\
\underline{-15} \\
\text{26} \\
\underline{-24} \\
\text{27} \\
\underline{-27} \\
\text{0}
\end{array}
\]

Then, round the quotient to the thousandths place.
The quotient of 10.0767 divided by 3 is 3.359.

**Example 5**

\[
0.48684 \div 2 = \\
\]

First, divide to find the quotient. You can stop at the ten-thousandths place.

\[
\begin{array}{c}
2) 0.48684 \\
-4 \\
-8 \\
-8 \\
-6 \\
-8 \\
-4 \\
\end{array}
\]

Then, round the quotient to the thousandths place.

\[
0.2434 \approx 0.243 \\
\]

The quotient of 0.48684 divided by 2 rounded to the thousandths is 0.243.

**Review**

Find each quotient rounded to the nearest thousandths.

1. 0.54686 ÷ 2
2. 0.84684 ÷ 2
3. 0.154586 ÷ 2
4. 0.34689 ÷ 3
5. 0.994683 ÷ 3
6. 0.154685 ÷ 5
7. 0.546860 ÷ 5
8. 0.25465 ÷ 5
9. 0.789003 ÷ 3
10. 0.18905 ÷ 5
11. 0.27799 ÷ 9
12. 0.354680 ÷ 10
13. 0.454686 ÷ 6
14. 0.954542 ÷ 2
15. 0.8546812 ÷ 4
3.9. Decimal Rounding and Division

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.12.

Resources

 MEDIA
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 MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/166304
In this concept, you will learn how to divide a decimal by a decimal.

Liam is almost done building a bookcase. He just needs to add the shelves. Each shelf must be 26.5 inches long. He has a piece of wood that is 90.1 inches long. How any shelves can he make from that piece of wood?

In this concept, you will learn how to divide a decimal by a decimal.

Dividing Decimals by Decimals

You can divide a decimal number by another decimal number, but placing the decimal back into the quotient can become tricky. Simplify the process by change the divisor to a whole number.

Here is a division problem.

\[ \frac{10.4}{2.6} = \]

The divisor is 2.6. To change a decimal number to a whole number, multiply the decimal number by a power of ten to move the decimal point. Multiply 2.6 by 10 to move the decimal point one space to the right.

\[ 2.6 \times 10 = 26 \]

Dividing by 26 is easier than dividing by 2.6. However, dividing 10.4 by 26 is not the same as dividing 10.4 by 2.6. If you change the divisor, you must also change the dividend.

Remember that a division problem can also be written as a fraction. The dividend is placed in the numerator and the divisor is placed in the denominator. In this problem, 10.4 is the dividend and is placed in the numerator. 2.6 is the divisor and is placed in the denominator.

\[ 10.4 \div 2.6 = \frac{10.4}{2.6} \]
If you change the denominator, you must also change the numerator the same way in order for the fraction to have the same value. Multiply the numerator and denominator by 10.

\[
10.4 \div 2.6 = \frac{10.4 \times 10}{2.6 \times 10} = \frac{104}{26}
\]

Now write the new fraction as a division problem and divide to find the quotient.

\[
104 \div 26 \quad \text{or} \quad 26 \overline{)104}
\]

\[
\begin{array}{c|c}
26 & 104 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\text{4} & 104 \\
\hline
\text{26} & 0
\end{array}
\]

10.4 ÷ 2.6 is the same as 104 ÷ 26. The quotient of 10.4 divided by 2.6 is 4.

Here is another division problem. This time the divisor has two decimal places.

\[
0.45 \overline{)1.44}
\]

First, change the divisor to a whole number by multiplying it by a power of ten. Multiply 0.45 by 100. Then, do the same thing to the dividend.

\[
\begin{align*}
0.45 \times 100 &= 45 \\
1.44 \times 100 &= 144
\end{align*}
\]

Here is the new division problem. Divide to find the quotient.

\[
\begin{array}{c|c}
3 & 144 \\
\hline
45 & -135 \\
\hline
\text{9}
\end{array}
\]

Use **zero placeholders** to continue dividing.

\[
\begin{array}{c|c}
3.2 & 144.0 \\
\hline
45 & -135 \\
\hline
90 & -90 \\
\hline
0
\end{array}
\]

The quotient of 1.44 divided by 0.45 is 3.2.
Notice the pattern. You move the decimal the same number of spaces in the dividend as you do in the divisor.

\[
\begin{align*}
2.6 & \rightarrow 2 \overset{3}{6} \quad 0.45 & \rightarrow 45 \\\\\\\\\ \quad 1.35 & \rightarrow 135 \\
10.4 & \rightarrow 10 \overset{1}{4} \quad & & \\
\end{align*}
\]

To multiply a decimal number by a decimal number, change the divisor to a whole number and move the decimal point the same number of spaces in the dividend. Then, divide to find the quotient.

**Examples**

**Example 1**

Earlier, you were given a problem about Liam and his bookcase.

He wants to know how many 26.5 inch shelves he can make from a piece of wood that is 90.1 inches long. Divide to find the answer.

\[
\begin{align*}
26.5 & ) 90.1 \\
\end{align*}
\]

First, change the divisor to a whole number.

\[
26.5 \rightarrow 26 \overset{3}{5},
\]

Then, do the same thing to the dividend.

\[
90.1 \rightarrow 90 \overset{1}{1},
\]

Next, write the new division problem and divide to find the quotient.

\[
\begin{align*}
3.4 \\
\hline
265 & ) 901.0 \\
\hline
& 795 \\
\hline
& 1060 \\
\hline
& 1060 \\
\hline
& 0
\end{align*}
\]

Liam can make at least 3 shelves from the piece of wood.

**Example 2**

Divide the decimals.

\[
\begin{align*}
3.45 & ) 7.245 \\
\end{align*}
\]
First, change 3.45 to a whole number by multiplying it by a power of ten. Multiply 3.45 by 100 or simply move the decimal point two spaces to the right.

\[3.45 \times 100 = 345\text{ or } \frac{345}{1}\]

Then, do the same thing to the dividend. Multiply 7.245 by 100 or move the decimal point two spaces to the right.

\[7.245 \times 100 = 724.5\text{ or } \frac{724.5}{1}\]

Next, write the new division problem and divide to find the quotient.

\[
\begin{array}{rcccc}
  & 2 & 1 \\
\hline
3 & 4 & 5 & ) & 7 & 2 & 4 & . & 5 \\
  & 6 & 9 & 0 \\
\hline
  & 3 & 4 & 5 \\
  & 3 & 4 & 5 \\
\hline
  & 0 \\
\end{array}
\]

The quotient of 7.245 divided by 3.45 is 2.1.

**Divide the decimals. Use zero placeholders if needed.**

**Example 3**

\[
\begin{array}{rcccc}
  & 4 & . & 8 \\
\hline
1 & 2 & ) & 4 & . & 8 \\
  & 4 & 8 \\
\hline
  & 0 \\
\end{array}
\]

First, change the divisor to a whole number.

\[1.2 \times 10 = 12\text{ or } \frac{12}{1}\]

Then, do the same thing to the dividend.

\[4.8 \times 10 = 48\text{ or } \frac{48}{1}\]

Next, write the new division problem and divide to find the quotient.

\[
\begin{array}{rcccc}
  & 4 \\
\hline
1 & 2 & ) & 4 & 8 \\
  & 4 & 8 \\
\hline
  & 0 \\
\end{array}
\]

The quotient of 4.8 divided by 1.2 is 4.
Example 4

\[ 5.6 \div 14.28 \]

First, change the divisor to a whole number.

\[ 5.6 \times 10 = 56 \quad \text{or} \quad 5 \frac{6}{10}. \]

Then, do the same thing to the dividend.

\[ 14.28 \times 10 = 142.8 \quad \text{or} \quad 14 \frac{2}{10}.8 \]

Next, write the new division problem and divide to find the quotient.

\[ \begin{array}{r} \text{2.55} \\ 56 \overline{) 142.80} \\ -112 \\ \hline \end{array} \]

\[ \begin{array}{r} \text{308} \\ -280 \\ \hline \end{array} \]

\[ \begin{array}{r} \text{280} \\ -280 \\ \hline \end{array} \]

\[ \begin{array}{r} \text{0} \end{array} \]

The quotient of 14.28 divided by 5.6 is 2.55.

Example 5

\[ 6.98 \div 13.96 \]

First, change the divisor to a whole number.

\[ 6.98 \times 100 = 698 \quad \text{or} \quad 698 \frac{}{10}. \]

Then, do the same thing to the dividend.

\[ 13.96 \times 100 = 1,396 \quad \text{or} \quad 1,396 \frac{}{10}. \]

Next, write the new division problem and divide to find the quotient.

\[ \begin{array}{r} \text{2} \\ 698 \overline{) 1396} \\ -1,396 \\ \hline \end{array} \]

\[ \begin{array}{r} \text{0} \end{array} \]

The quotient of 13.96 divided by 6.98 is 2.
3.10. Division of Decimals by Decimals

Review

Divide the following decimals. Use zero place holders if needed.

1. 1.2)4.08
2. 3.5)12.6
3. 14.5)29
4. 8.9)11.57
5. 0.32)0.08
6. 1.2)7.8
7. 9.6)11.52
8. 14.5)33.35
9. 12.5)7.5
10. 2.5)13.29
11. 1.2)7.2
12. 0.8)1.8
13. 4.6)10.58
14. 1.6)0.3

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.16.

Resources

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3.11 Factor Pairs

In this concept, you will learn how to identify factor pairs.

The 6th grade class is going on a field trip to the state fair. There are a total of 156 students on the trip. The teachers are trying to decide on how to evenly group the students so the groups are not too small or too large. How many different combinations of equal groups can they make with 156 students? What combination would produce a group size of around 15 students per group?

In this concept, you will learn how to identify factor pairs.

Finding Factor Pairs

A factor is a number or a group of numbers that are multiplied together to make a product. Two factors multiplied together for a product is called a factor pair.

Find the factor pairs for 12.

First, find all the factors for 12 starting with 1. Any number multiplied by 1 is that number. Therefore, one is a factor of every number.

\[ 1 \times 12 = 12 \]

After starting with 1, move on to 2, then 3 and so on until you have listed out all of the factors for 12. Knowing the multiplication facts is useful for finding factor pairs.

\[ 1 \times 12 = 12 \\
2 \times 6 = 12 \\
3 \times 4 = 12 \]

Then, list the factor pairs for 12.

12 – 1 and 12, 2 and 6, 3 and 4
A factor is also a number that will divide evenly into another number, so you can find factor pairs by dividing as well. This is useful when finding the factors of larger numbers.

Find the factor pairs for 72
First, find all the factors for 72. Divide 72 starting with the number 1, and continue on until you have found all of the factors that divide into 72 evenly.

\[
\begin{align*}
72 \div 1 &= 72 \\
72 \div 2 &= 36 \\
72 \div 3 &= 24 \\
72 \div 4 &= 18 \\
72 \div 6 &= 12 \\
72 \div 8 &= 9 \\
\end{align*}
\]

Remember that you only need to find one factor pair once. 72 ÷ 8 = 9 and 72 ÷ 9 = 8 are in the same fact family.
Then, list the factor pairs for 72.

72 – 1 and 71, 2 and 36, 3 and 24, 4 and 18, 6 and 12, 8 and 9

Examples

Example 1
Earlier, you were given a problem about the 6th grade field trip to the State Fair.
The teachers are trying to figure out how to evenly group 156 students. Find all the possible group combinations using factor pairs and the combination that closely produces around 15 students per group.
First, find the factor pairs for 156 students.

\[
\begin{align*}
1 \times 156 &= 156 \\
2 \times 76 &= 156 \\
3 \times 52 &= 156 \\
4 \times 39 &= 156 \\
6 \times 26 &= 156 \\
12 \times 13 &= 156 \\
\end{align*}
\]

Then, find total number of group combinations. There are 6 factor pairs, but there are 2 possible combinations per factor pair. For example, the factor pair 1 and 156 can make 1 group of 156 students or 156 groups of 1 student.

6 × 2 = 12

There are 12 possible combinations of groups. The factor pair that produces around 15 students is 12 and 13, 12 groups of 13 students or 13 groups of 12 students.

Example 2
List the factor pairs of 18.
First, find the factor pairs for 18.

\[
\begin{align*}
1 \times 18 &= 18 \\
2 \times 9 &= 18 \\
3 \times 6 &= 18 \\
\end{align*}
\]
Then, list the factor pairs for 18.

\[ 18 - 1 \text{ and } 18, 2 \text{ and } 9, 3 \text{ and } 6 \]

**Example 3**

List the factor pairs for the following number.

\[ 36 \]

First, find the factor pairs for 36.

\[ 1 \times 36 = 39 \quad 4 \times 9 = 39 \]
\[ 2 \times 18 = 39 \quad 6 \times 6 = 39 \]
\[ 3 \times 12 = 39 \]

Then, list the factor pairs for 36.

\[ 36 - 1 \text{ and } 36, 2 \text{ and } 18, 3 \text{ and } 12, 4 \text{ and } 9, 6 \text{ and } 6 \]

**Example 4**

List the factor pairs for the following number.

\[ 24 \]

First, find the factor pairs for 24.

\[ 1 \times 24 = 3 \times 8 \]
\[ 2 \times 12 = 4 \times 6 \]

Then, list the factor pairs for 24.

\[ 24 - 1 \text{ and } 24, 2 \text{ and } 12, 3 \text{ and } 8, \text{ and } 4 \text{ and } 6 \]

**Example 5**

List the factor pairs for the following number.
First, find the factor pairs for 90.

\[
\begin{align*}
90 \div 1 &= 72 \\
90 \div 2 &= 45 \\
90 \div 3 &= 30 \\
90 \div 5 &= 18 \\
90 \div 6 &= 15 \\
90 \div 9 &= 10
\end{align*}
\]

Then, list the factor pairs for 90.

90 – 1 and 90, 2 and 45, 3 and 30, 5 and 18, 6 and 15, 9 and 10

**Review**

List the factor pairs for each of the following numbers.

1. 12
2. 10
3. 15
4. 16
5. 56
6. 18
7. 20
8. 22
9. 23
10. 25
11. 27
12. 31
13. 81
14. 48
15. 24
16. 30

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 5.1.
Richard is making gift bags. He has 36 pencils and 28 pens. How many gift bags can Richard make if there are the same number of pencils and pens in each bag? Use factor trees to solve this problem. How many pencils and pens will be in each bag?

In this concept, you will learn to find the greatest common factor using factor trees.

Finding the Greatest Common Factor Using Factor Trees

The greatest common factor (GCF) is the greatest factor that two or more numbers have in common. The GCF can be found by making a list and comparing all the factors. A factor tree can also be used to find the GCF. The GCF is the product of the common prime factors.

Let’s find the GCF of 20 and 30 using a factor tree.

First, make a factor tree for each number.

20
/ \
4 \times 5
/ \n2 \times 2

30
/ \
5 \times 6
/ \n2 \times 3

Then, identify the common factors. The numbers 20 and 30 have the factors 2 and 5 in common.
3.12. Greatest Common Factor Using Factor Trees

\[
20 = 2 \times 2 \times 5 \quad 30 = 2 \times 3 \times 5
\]

Next, multiply the common factors to find the GCF. If there is only one common factor, there is no need to multiply.

\[
2 \times 5 = 10
\]

The GCF of 20 and 30 is 10.

Note that if the numbers being compared have no factors in common using a factor tree, they still have the factor 1 in common.

**Examples**

**Example 1**

Earlier, you were given a problem about Richard who needs to make gift bags with 36 pencils and 28 pens. Use factor trees to find the most number of bags he can make that have the same number of pencils and pens in each.

First, make a factor tree for each number.

\[
\begin{align*}
36 &= 2 \times 2 \times 3 \times 3 \\
28 &= 2 \times 2 \times 7
\end{align*}
\]

Then, identify the common factors. The common factors are two 2s.

\[
\begin{align*}
36 &= 2 \times 2 \times 3 \times 3 \\
28 &= 2 \times 2 \times 7
\end{align*}
\]

Next, multiply to common factors to find the GCF.

\[
2 \times 2 = 4
\]

Finally, divide the number of pencils and pens by the GCF, 4.

\[
\begin{align*}
\text{pencils} &= \frac{36}{4} = 9 \\
\text{pens} &= \frac{28}{4} = 7
\end{align*}
\]

Richard can make 4 gift bags that have 9 pencils and 7 pens in each bag.
Example 2

Find the GCF of 36 and 54 using factor trees.
First, make a factor tree for each number.

\[
\begin{align*}
36 &= 2 \times 2 \times 3 \times 3 \\
54 &= 2 \times 3 \times 3 \times 3
\end{align*}
\]

Then, identify the common factors. The numbers 36 and 54 have the factors 2 and two 3s in common.

\[
\begin{align*}
36 &= 2 \times 2 \times 3 \times 3 \\
54 &= 2 \times 3 \times 3 \times 3
\end{align*}
\]

Next, multiply the common factors to find the GCF.

\[
2 \times 3 \times 3 = 18
\]

The GCF of 36 and 54 is 18.

Example 3

Find the greatest common factor using factor trees.

14 and 28

First, make a factor tree for each number.
3.12. Greatest Common Factor Using Factor Trees

Then, identify the common factors. The numbers 14 and 28 have the factors 2 and 7 in common.

\[
\begin{align*}
14 &= 2 \times 7 \\
28 &= 2 \times 2 \times 7
\end{align*}
\]

Next, multiply the common factors to find the GCF.

\[
2 \times 7 = 14
\]

The GCF of 14 and 28 is 14.

Example 4

Find the greatest common factor using factor trees.

24 and 34

First, make a factor tree for each number.
Then, identify the common factors. The numbers 24 and 34 have the factor 2 in common.

\[
24 = 2 \times 2 \times 2 \times 3 \quad 34 = 2 \times 17
\]

The GCF of 12 and 24 is 12.

**Example 5**

Find the greatest common factor using factor trees.

19 and 63

First, make a factor tree for each number.

\[
19 = 1 \times 19 \quad 63 = 3 \times 3 \times 7
\]
Then, identify the common factors. The numbers 19 and 63 have the factor 1 in common.

\[
19 = 1 \times 19 \\
63 = 3 \times 3 \times 7
\]

The GCF of 19 and 63 is 1.

**Review**

Find greatest common factor for each pair of numbers.

1. 14 and 28
2. 14 and 30
3. 16 and 36
4. 24 and 60
5. 72 and 108
6. 18 and 81
7. 80 and 200
8. 99 and 33
9. 27 and 117
10. 63 and 126
11. 89 and 178
12. 90 and 300
13. 56 and 104
14. 63 and 105
15. 72 and 128

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 5.6.

**Resources**

**MEDIA**

Click image to the left or use the URL below.
URL: [https://www.ck12.org/flx/render/embeddedobject/162149](https://www.ck12.org/flx/render/embeddedobject/162149)

**MEDIA**

Click image to the left or use the URL below.
URL: [https://www.ck12.org/flx/render/embeddedobject/162151](https://www.ck12.org/flx/render/embeddedobject/162151)
Mara is in making flower arrangements for a party. She has 48 carnations and 42 daisies. She wants there to be an equal number of carnations and daisies in each bouquet. What is the most number of bouquets she can make? How many of each flower will they contain?

In this concept, you will learn to find the greatest common factors of numbers using lists.

Finding The Greatest Common Factor Using Lists

The greatest common factor (GCF) is the greatest factor that two or more numbers have in common. One way to find the GCF is to make lists of the factors for two numbers and then choose the greatest factor that the two factors have in common.

Find the GCF for 12 and 16. It is helpful to order them from smallest to largest in order to make sure that you cover every factor.

First, find all the factors of 12 and 16 and write them in a list in the order of least to greatest.

12 — 1, 2, 3, 4, 6, 12
16 — 1, 2, 4, 8, 16

One way to check if all the factors are listed is to use the rainbow method. Draw a line from one part of a factor pair to the other. The resulting image should resemble a rainbow.
3.13. Greatest Common Factor Using Lists

Next, identify the GCF, the largest number that appears in both lists. The GCF for 12 and 16 is 4.

**Examples**

**Example 1**

Earlier, you were given a problem about Mara and her flowers.

Mara has 48 carnations and 42 daisies and wants each bouquet to have the same number of flowers. Compare the factors 48 and 42 and find the greatest common factor.

First, find all the factors of 48 and 42 and write them from least to greatest.

\[
48 - 1, 2, 3, 4, 6, 12, 16, 24, 48 \\
42 - 1, 2, 3, 6, 7, 14, 21, 42
\]

Then, identify the GCF. The GCF for 48 and 42 is 6.

Next, find the number of carnations and daisies in 6 bouquets.

\[
carnations : 48 \div 6 = 8 \\
daisies : 42 \div 6 = 7
\]

The most number of bouquets Mara can make will be 6. Each will have 8 carnations and 7 daisies.

**Example 2**

What is the GCF of 140 and 124?

First, find all the factors of 140 and 124 and write them in a list in the order of least to greatest.
Next, identify the GCF, the largest number that appears in both lists. The GCF for 140 and 124 is 4.

**Example 3**

Find the GCF for the pair of numbers.

24 and 36

First, find all the factors of 24 and 36 and write them in a list in the order of least to greatest.

\[24 - 1, 2, 3, 4, 6, 8, 12, 24\]

\[36 - 1, 2, 3, 4, 6, 9, 12, 18, 36\]

Next, identify the GCF, the largest number that appears in both lists. The GCF for 24 and 36 is 12.

**Example 4**

Find the GCF for the pair of numbers.

10 and 18

First, find all the factors of 10 and 18 and write them in a list in the order of least to greatest.

\[10 - 1, 2, 5, 10\]

\[18 - 1, 2, 3, 6, 9, 18\]

Next, identify the GCF, the largest number that appears in both lists. The GCF for 10 and 18 is 2.

**Example 5**

Find the GCF for the pair of numbers.

18 and 45

First, find all the factors of 18 and 45 and write them in a list in the order of least to greatest.
18 – 1, 2, 3, 6, 9, 18
45 – 1, 3, 5, 9, 25, 45

Next, identify the GCF, the largest number that appears in both lists. The GCF for 18 and 45 is 9.

Review

Find the GCF for each pair of numbers.

1. 9 and 21
2. 4 and 16
3. 6 and 8
4. 12 and 22
5. 24 and 30
6. 35 and 47
7. 35 and 50
8. 44 and 121
9. 48 and 144
10. 60 and 75
11. 21 and 13
12. 14 and 35
13. 81 and 36
14. 90 and 80
15. 22 and 33
16. 11 and 13
17. 15 and 30
18. 28 and 63
19. 67 and 14
20. 18 and 36

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.5.

Resources
MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/162151
3.14 Divisibility Rules to Find Factors

In this concept, you will learn and apply the divisibility rules to find factors of given numbers.

Lisa, Mark, and Stacy are meeting up for lunch. Their total comes out to $28.87. They decide they want to split the bill between the three of them. Will they be able to split the check equally? And how much will each of them pay?

In this concept, you will learn and apply the divisibility rules to find factors of given numbers.

Finding Factors by Using Divisibility Rules

There are some quick tests you can use to see if a large number is divisible by another number.

Divisibility rules help determine if a number is divisible by let’s say 2 or 3 or 4. This can help us to identify the factors of a number. Here is a chart that shows all of the basic divisibility rules.
Some of these rules will be more useful than others, but this chart will help you.

Find a factor of 1,346 using the divisibility rules. Go through each rule and see if it applies.

1. The last digit is even—this number is divisible by 2.
2. The sum of all the digits is 14—this number is not divisible by 3.
3. The last two digits are not divisible by 4—this number is not divisible by 4.
4. The last digit is not zero or five—this number is not divisible by 5.
5. 1,346 – 12 = 1,334—this number is not divisible by 7.
6. The last three numbers are not divisible by 8.
7. The sum of the digits is 14—this number is not divisible by 9.
8. The number does not end in zero—this number is not divisible by 10.
9. The number is not divisible by 3 and 4.
10. The number 1,346 is divisible by 2.

**Examples**

**Example 1**

Earlier, you were given a problem about Lisa and her friends having lunch. They want to split a bill of $28.87 equally between the three of them. Check the divisibility rule and divide to see how much each of them will pay.

First, check to see if the sum of all the digits is divisible by 3.

\[ 2 + 8 + 8 + 7 = 25 \rightarrow \text{no} \]

Then, divide the total by 3.
$28.87 \div 3 = 9.623333\ldots$

$28.87$ is not divisible by $3$. Two people will pay $9.62$ and one person will pay $9.63$.

**Example 2**

Test if $918$ divisible by $9$. Why or why not?
To figure this out, use the divisibility rules. Check to see if the sum of the digits is divisible by $9$.

\[9 + 1 + 8 = 18\]

$18$ is divisible by $9$, therefore $918$ is also divisible by $9$.

**Example 3**

Use the divisibility rules to answer the following question.
Is $3,450$ divisible by $10$?
First, check to see if the number ends in $0$.

$3,450 \rightarrow yes$

$3,450$ is divisible by $0$.

**Example 4**

Use the divisibility rules to answer the following question.
Is $1,298$ divisible by $3$?
First, check if the sum of all digits is divisible by $3$.

\[1 + 2 + 9 + 8 = 20 \rightarrow no\]

$1,298$ is not divisible by $3$.

**Example 5**

Use the divisibility rules to answer the following question.
Is $3,678$ divisible by $2$?
First, check if the last digit is even.

$3,678 \rightarrow yes$

$3,678$ is divisible by $2$. 

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Review

Use the divisibility rules to answer the following questions. Explain your reasoning.

1. Is 18 divisible by 3?
2. Is 22 divisible by 2?
3. Is 44 divisible by 6?
4. Is 112 divisible by 2 and 3?
5. Is 27 divisible by 9 and 3?
6. Is 219 divisible by 9?
7. Is 612 divisible by 2 and 3?
8. Is 884 divisible by 4?
9. Is 240 divisible by 5?
10. Is 782 divisible by 7?
11. Is 212 divisible by 4 and 6?
12. Is 456 divisible by 6 and 3?
13. Is 1848 divisible by 8 and 4?
14. Is 246 divisible by 2?
15. Is 393 divisible by 3?
16. Is 7450 divisible by 10?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.2.

Resources

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/166547
3.15 Least Common Multiple

In this concept, you will learn to find the least common multiples of numbers by using lists.

Arjay is planning a barbecue. He is at the store to buy hot dogs and hot dog buns. The hot dogs come in packs of 8 while the buns come in packs of 6. At least how many packages of each should he buy to have the same number of hot dogs and buns?

In this concept, you will learn to find the least common multiples of numbers by using lists.

Finding the Least Common Multiple Using Lists

Common multiples are multiples that two or more numbers have in common. The least common multiple (LCM) is the smallest multiple that two numbers have in common.

Let’s look back at the common multiples for 3 and 4.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36
4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48

Some of the common multiples of 3 and 4 are 12, 24 and 36. The LCM of these two numbers is 12. It is the smallest number that they both have in common.

To find the LCM, list the multiples of the two numbers. Stop when you have found the first common multiple.

Examples

Example 1

Earlier, you were given a problem about Arjay on his shopping trip.
Arjay wants to know at least how many packages of 8 hot dogs and 6 hot dog buns he needs to buy to get the same amount of each. Find the LCM of 8 and 6 to find the number of packages for each.

First, list the multiples of both 8 and 6. Stop at the first common multiple.

8 = 8, 16, 24, 32
6 = 6, 12, 18, 24

The LCM of 8 and 6 is 24.

Then, find number of packages of hot dogs and buns needed to make 24 hotdogs.

24 ÷ 8 = 3
24 ÷ 6 = 4

Arjay must buy 3 packages of hotdogs and 4 packages of hotdog buns to get the same number of each.

Example 2

Find the LCM of 20 and 15.

First, list out the multiples of both 20 and 15. Stop at the first common multiple.

20 = 20, 40, 60, 80, 100
15 = 30, 45, 60

The LCM of 20 and 15 is 60.

Example 3

Find the LCM of the pair of numbers.

5 and 3

First, list the multiples of both 5 and 3. Stop at the first common multiple.

5 = 5, 10, 15, 20, 25
3 = 3, 6, 9, 12, 15

The LCM of 5 and 3 is 15.

Example 4

Find the LCM of the pair of numbers.

2 and 6

First, list the multiples of both 2 and 6. Stop at the first common multiple.

2 = 2, 4, 6, 8
6 = 6

The LCM of 2 and 6 is 6.

Example 5

Find the LCM of the pair of numbers.
3.15. Least Common Multiple

4 and 6
First, list the multiples of both 4 and 6. Stop at the first common multiple.
4 = 4, 8, **12**, 16, 20
6 = 6, **12**
The LCM of 4 and 6 is 12.

**Review**

Find the LCM of each pair of numbers.

1. 3 and 5
2. 2 and 3
3. 3 and 4
4. 2 and 6
5. 3 and 9
6. 5 and 7
7. 4 and 12
8. 5 and 6
9. 10 and 12
10. 5 and 8

**Review (Answers)**

To see the Review answers, open this [PDF file](https://www.ck12.org/flx/render/embeddedobject/166632) and look for section 5.10.

**Resources**

[Media](https://www.ck12.org/flx/render/embeddedobject/166632)
In this concept, you will learn how to compare fractions using the lowest common denominator.

Comparing Fractions Using the Lowest Common Denominator

Some fractions have different denominators, the bottom number of a fraction. The numerator refers to the top number of a fraction.

Here are two fractions with different denominators.

\[
\frac{1}{4} \text{ and } \frac{2}{3}
\]

Remember that the denominator is the number of parts the whole has been divided into. In the first fraction, one-fourth, the whole has been divided into four parts. The second fraction, two-thirds, has been divided into three parts. In this example, you cannot compare the numerators because the parts of each fraction have different values.

You use greater than (＞), less than (＜), or equal to (=) to compare two fractions. It is easy to compare fractions with the same denominator.

Compare these two fractions.

\[
\frac{1}{5} \text{ and } \frac{3}{5}
\]

Both fractions represent a whole that is divided into 5 parts. If the fractions were pizzas that were divided into 5 parts, one-fifth of a pizza would be less than with three-fifths of the same pizza. Therefore, you can compare those fractions like this.
To compare fractions different denominators, rewrite the fractions so they have a common denominator.
Let’s compare the two fractions from earlier.

\[
\frac{1}{4} < \frac{2}{3}
\]

Rewrite the denominators by finding the least common multiple of each denominator. Remember that the least common multiple (LCM) is the smallest multiple that two numbers have in common. This LCM becomes the lowest common denominator (LCD).

First, list the multiples for 3 and 4 and find the LCM.

\[
\begin{align*}
4, & \quad 8, \quad 12, \quad 16 \\
3, & \quad 6, \quad 9, \quad 12
\end{align*}
\]

The LCM for 3 and 4 is 12.

Then, rewrite each fraction in terms of twelfths. Make a fraction equivalent to one-fourth in terms of twelfths, and make a fraction equivalent to two-thirds in terms of twelfths.

\[
\frac{1}{4} = \frac{3}{12}
\]

To make equivalent fractions, multiply or divide the numerator and the denominator by the same number to create the equal fraction. 4 is multiplied by 3 to get 12. Complete the equivalent fraction by also multiplying the numerator by 3.

\[
\frac{1}{4} = \frac{3}{12}
\]

Now work on rewriting two-thirds in terms of twelfths. 3 is multiplied by 4 to get 12. Multiply the numerator by 4.

\[
\frac{2}{3} = \frac{8}{12}
\]

Next, compare the fractions now that both fractions have been written in terms of twelfths.

\[
\frac{3}{12} < \frac{8}{12}
\]

so

\[
\frac{1}{4} < \frac{2}{3}
\]

\[\frac{1}{4}\] is less than \[\frac{2}{3}\].

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Examples

Example 1

Earlier, you were given a problem about the ice cream social.
Terrence and Emilia estimated that one-third of the class will want to eat vanilla ice cream and four-sevenths of the class will want to eat chocolate ice cream. Compare the fractions to see which flavor will be more popular.

\[
\frac{1}{3} \quad \frac{4}{7}
\]

First, find the LCM of 3 and 7. The lowest common denominator will be 21.
Then, rewrite each fraction with the lowest common denominator. Multiply the numerator and denominator of \(\frac{1}{3}\) by 7. Multiply the numerator and denominator of \(\frac{4}{7}\) by 3.

\[
\frac{1}{3} = \frac{7}{21}
\]
\[
\frac{4}{7} = \frac{12}{21}
\]

Next, compare the equivalent fractions.

\[
\frac{7}{21} < \frac{12}{21}
\]

Chocolate will be more popular.

Example 2

Rewrite each fraction with the lowest common denominator and compare using <, >, or =.

\[
\frac{6}{9} \quad \frac{3}{4}
\]

First, find the LCM of 9 and 4. The lowest common denominator will be 36.

9 = 9, 18, 27, 36
4 = 4, 12, 16, 20, 24, 28, 32, 36

Then, rewrite each fraction with the lowest common denominator. Multiply the numerator and denominator of \(\frac{6}{9}\) by 4. Multiply the numerator and denominator of \(\frac{3}{4}\) by 9.

\[
\frac{6}{9} = \frac{24}{36}
\]
\[
\frac{3}{4} = \frac{27}{36}
\]

Next, compare the equivalent fractions.

\[
\frac{24}{36} < \frac{27}{36}
\]

\(\frac{6}{9}\) is less than \(\frac{3}{4}\).
Example 3

Compare the fractions.

\[
\frac{2}{5} \quad \frac{6}{10}
\]

First, find the LCM of 5 and 10. The lowest common denominator will be 10.

Then, rewrite each fraction with the lowest common denominator. Multiply the numerator and denominator of \(\frac{2}{5}\) by 2. \(\frac{6}{10}\) is already a fraction of tenths.

\[
\frac{2}{5} = \frac{4}{10}
\]

Next, compare the equivalent fractions.

\[
\frac{4}{10} < \frac{6}{10}
\]

\(\frac{2}{5}\) is less than \(\frac{6}{10}\).

Example 4

Compare the fractions.

\[
\frac{2}{3} \quad \frac{1}{9}
\]

First, find the LCM of 3 and 9. The lowest common denominator will be 9.

Then, rewrite each fraction with the lowest common denominator. Multiply the numerator and denominator of \(\frac{2}{3}\) by 3. \(\frac{1}{9}\) does not change.

\[
\frac{2}{3} = \frac{6}{9}
\]

Next, compare the equivalent fractions.

\[
\frac{6}{9} > \frac{1}{9}
\]

\(\frac{2}{3}\) is greater than \(\frac{1}{9}\).
Example 5

Compare the fractions.

\[
\frac{3}{4} \quad \frac{6}{8}
\]

First, find the LCM of 4 and 8. The lowest common denominator will be 8.

Then, rewrite each fraction with the lowest common denominator. Multiply the numerator and denominator of \(\frac{3}{4}\) by 2. \(\frac{6}{8}\) does not change.

\[
\frac{3}{4} = \frac{6}{8}
\]

Next, compare the equivalent fractions.

\[
\frac{6}{8} = \frac{6}{8}
\]

\(\frac{3}{4}\) is equal to \(\frac{6}{8}\).

Review

Rename each in terms of tenths.

1. \(\frac{1}{5}\)
2. \(\frac{3}{5}\)
3. \(\frac{1}{2}\)
4. \(\frac{4}{5}\)

Complete each equal fraction.

5. \(\frac{1}{5} = \frac{5}{25}\)
6. \(\frac{2}{5} = \frac{10}{25}\)
7. \(\frac{5}{6} = \frac{25}{30}\)
8. \(\frac{2}{7} = \frac{10}{35}\)
9. \(\frac{4}{9} = \frac{20}{45}\)
10. \(\frac{3}{8} = \frac{30}{40}\)

Identify the lowest common multiple for each pair of numbers.

11. 3 and 6
12. 4 and 10
13. 5 and 3
14. 7 and 2
15. 8 and 4
3.16. Fraction Comparison with Lowest Common Denominators

16. 6 and 4
17. 8 and 5
18. 12 and 5
19. 9 and 2
20. 6 and 7

Compare the following fractions using <, >, or =

21. $\frac{1}{2}$ ___ $\frac{1}{3}$
22. $\frac{2}{3}$ ___ $\frac{3}{9}$
23. $\frac{4}{6}$ ___ $\frac{2}{3}$
24. $\frac{6}{18}$ ___ $\frac{4}{5}$
25. $\frac{9}{18}$ ___ $\frac{3}{6}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.11.
In this concept, you will learn to order fractions using lowest common denominators.

Sam surveyed his classmates for a class assignment. He asked his classmates about some of their interests and hobbies. The results of the survey were:

\( \frac{7}{8} \) watched TV
\( \frac{1}{4} \) rode bikes
\( \frac{5}{8} \) read books
\( \frac{3}{4} \) played sports
\( \frac{3}{8} \) played an instrument or sang

Sam is using this information to make a presentation. What is the order of activities if Sam were to rank them from the most popular to least?

In this concept, you will learn to order fractions using lowest common denominators.

**Using the Least Common Denominator to Order Fractions**

Sometimes, you will need to write fractions in order from least to greatest or from greatest to least. This becomes very simple if the fractions have the same denominator.

Write in order from least to greatest.

\[
\frac{4}{9}, \frac{2}{9}, \frac{8}{9}, \frac{3}{9}, \frac{6}{9}
\]

Since all of these fractions have a common denominator, use the numerators and arrange them in order from the smallest numerator to the largest numerator.

The answer is \( \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{6}{9}, \frac{8}{9} \)

To order fractions that do not have a common denominator, rewrite all of the fractions using the **lowest common denominator** (LCD).
Order these fractions from least to greatest.

\[
\begin{align*}
\frac{2}{3}, & \frac{1}{4}, \frac{1}{2}, \frac{5}{6} \\
\end{align*}
\]

First, find the LCD. The lowest common multiple of 3, 4, 2, and 6 is 12. Remember, if you cannot figure out the LCD in your head, list the multiples to find the least common multiple (LCM).

\[
\begin{align*}
3 - 3, 6, 9, 12 & \quad 4 - 4, 8, 12 & \quad 2 - 2, 4, 6, 8, 10, 12 & \quad 6 - 6, 12 \\
\end{align*}
\]

Then, rewrite each fraction with the denominator 12.

\[
\begin{align*}
\frac{2}{3} &= \frac{8}{12} \\
\frac{1}{4} &= \frac{3}{12} \\
\frac{1}{2} &= \frac{6}{12} \\
\frac{5}{6} &= \frac{10}{12} \\
\end{align*}
\]

Next, order the fractions from least to greatest.

\[
\begin{align*}
\frac{1}{4}, & \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \\
\end{align*}
\]

**Examples**

**Example 1**

Earlier, you were given a problem about Sam’s survey.

Sam wants to find the order of activities from most popular to least popular. Compare the fractions and list them from greatest to least.

\[
\begin{align*}
\frac{7}{8} & \text{ watched TV} \\
\frac{3}{4} & \text{ rode bikes} \\
\frac{5}{8} & \text{ read books} \\
\frac{3}{4} & \text{ played sports} \\
\frac{3}{8} & \text{ played an instrument or sang} \\
\end{align*}
\]

First, find the LCD. The LCD of 4 and 8 is 8.

Then, rewrite each fraction with the denominator 8. Three of the fractions already have denominators of 8. Find the equivalent fraction for \(\frac{3}{4}\) and \(\frac{3}{8}\).

\[
\begin{align*}
\frac{3}{4} &= \frac{6}{8} \\
\frac{3}{8} &= \frac{3}{8} \\
\end{align*}
\]

Next, order the fractions from greatest to least.

\[
\begin{align*}
\frac{7}{8}, & \frac{6}{8}, \frac{5}{8}, \frac{4}{8}, \frac{3}{8} \\
\end{align*}
\]
Sam should list the activities as:

Most Popular to Least Popular Activity

1. Watch TV
2. Play sports
3. Read books
4. Ride bike
5. Play instrument or sing

**Example 2**

Write the following fractions in order from least to greatest.

\[
\frac{4}{7}, \frac{2}{3}, \frac{5}{7}
\]

First, find the lowest common denominator. The lowest common multiple of 3 and 7 is 21.

Then, rewrite each fraction with the denominator 21.

\[
\frac{4}{7} = \frac{12}{21}, \quad \frac{2}{3} = \frac{14}{21}, \quad \frac{5}{7} = \frac{15}{21}
\]

Next, order the fractions from least to greatest.

\[
\frac{4}{7}, \frac{2}{3}, \frac{5}{7}
\]

Notice that the original order was in order from least to greatest.

**Example 3**

What would be the LCD for fractions with the denominators of 3, 5, and 6?

\[
3 \text{ – } 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, \quad 5 \text{ – } 5, 10, 15, 20, 25, 30, \quad 6 \text{ – } 6, 12, 18, 24, 30
\]

The LCD would be 30.

**Example 4**

Rewrite the fractions with a common denominator.

\[
\frac{4}{5}, \frac{1}{5}, \frac{2}{3}
\]
Example 5

Write the fractions above in order from greatest to least.

\[
\begin{align*}
\frac{4}{5} &= \frac{12}{15} \\
\frac{2}{3} &= \frac{10}{15} \\
\frac{1}{5} &= \frac{3}{15}
\end{align*}
\]

Review

Write each series in order from least to greatest.

1. \(\frac{5}{6}, \frac{1}{3}, \frac{4}{9}\)
2. \(\frac{6}{7}, \frac{1}{4}, \frac{2}{7}\)
3. \(\frac{6}{7}, \frac{4}{5}, \frac{2}{7}\)
4. \(\frac{1}{3}, \frac{3}{2}, \frac{2}{3}\)
5. \(\frac{5}{7}, \frac{3}{6}, \frac{3}{5}\)
6. \(\frac{1}{6}, \frac{2}{5}, \frac{3}{6}\)
7. \(\frac{4}{15}, \frac{4}{3}, \frac{3}{4}\)
8. \(\frac{10}{15}, \frac{5}{3}, \frac{3}{5}\)
9. \(\frac{4}{3}, \frac{1}{2}, \frac{4}{9}\)
10. \(\frac{7}{11}, \frac{7}{13}, \frac{3}{4}\)
11. \(\frac{4}{17}, \frac{3}{8}, \frac{1}{3}\)
12. \(\frac{6}{9}, \frac{1}{2}, \frac{4}{9}\)
13. \(\frac{7}{9}, \frac{3}{4}, \frac{1}{2}\)
14. \(\frac{6}{8}, \frac{3}{4}, \frac{2}{1}\)
15. \(\frac{1}{9}, \frac{7}{9}, \frac{7}{9}\)

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.12.

Resources
3.18 Prime Factorization

In this concept, you will learn to write the prime factorization of given numbers using a factor tree.

Connor is working on prime factorization for his math homework. He needs to find the prime numbers that, when multiplied together, produce the number 82. How can Connor complete this problem?

In this concept, you will learn to write the prime factorization of given numbers using a factor tree.

**Prime Factorization Using Factor Trees**

When a number is factored, it is broken down into two factors that are either prime numbers or composite numbers. **Prime numbers** are numbers that have only two factors, one and itself, and **composite numbers** are numbers that have more than two factors. Some examples of prime numbers are 2, 3, 11, etc. **Prime factorization** is the process of breaking down a number into a product of all prime numbers.

Here is a composite number.

36

The number 36 can be factored several different ways, but let’s factor it with $6 \times 6$.

$$36 = 6 \times 6$$

These two factors are not prime factors. Therefore, both factors can be factored again.

$$36 = 6 \times 6 = 2 \times 3 \times 2 \times 3$$

2 and 3 are both prime numbers.

One way to organize the factors is using a **factor tree**.
The number is written at the top of the factor tree. Then it is broken down into a factor pair, $6 \times 6$. $6$ can further be factored so the factor pairs are written underneath the $6$. Each number is continued to be factored until the factors are all prime numbers. Note that $36$ is written as a product of its primes at the bottom of the factor tree. Write the $2$s together and the $3$s together. Grouping like factors will help keep track of them.

The prime factorization is written using **exponential notation**, a method of writing repeated multiplication.

$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$

The base is the number being repeated and the exponent is the number of times the number is being multiplied. $2$ times $2$ is written as base $2$ with the exponent $2$, the number of times $2$ is multiplied by itself. $3$ times $3$ is written as base $3$ with an exponent of $2$.

**Examples**

**Example 1**

Earlier, you were given a problem about Connor’s prime factorization math problem. Connor needs to find the prime factorization of $82$. Use a factor tree to solve this problem.

First, start with $82$ at the top of the factor tree.

Then, begin by factoring $82$ using factor pairs.

$$82 = 2 \times 41$$

Connor finds that $82$ is the product of only $2$ prime numbers, $2$ and $41$.

**Example 2**

Write the prime factorization of $25$.

First, start with $25$ at the top of your factor tree.

Then, factor $25$ into the product of all prime numbers. $25$ can be factored into $5$ times $5$. $5$ is a prime number so this tells you that you have reached the bottom of the factor tree.
3.18. Prime Factorization

\[
\begin{align*}
25 \\
\quad \downarrow \\
5 \times 5 \\
\end{align*}
\]

\[25 = 5 \times 5\]

Next, write the factors in exponential notation. The base is 5 and the exponent is 2.

\[25 = 5^2\]

The prime factorization of 25 is \(5^2\).

**Example 3**

Write the prime factorization for the following number.

48

First, start with 48 at the top of your factor tree. Then, factor 48 into the product of all prime numbers.

\[
\begin{align*}
48 \\
\quad \downarrow \\
4 \times 12 \\
\quad \downarrow \\
2 \times 2 \times 6 \\
\quad \downarrow \\
2 \times 3 \\
\end{align*}
\]

\[48 = 2 \times 2 \times 2 \times 2 \times 3\]

Next, write the factors in exponential notation.

\[48 = 2^4 \times 3\]

The prime factorization of 48 is \(2^4 \times 3\).

**Example 4**

Write the prime factorization for the following number.
First, start with 100 at the top of your factor tree.
Then, factor 100 into the product of all prime numbers.

\[
100 = 2 \times 2 \times 5 \times 5
\]

Next, write the factors in exponential notation.

\[
100 = 2^2 \times 5^2
\]

The prime factorization of 100 is \(2^2 \times 5^2\).

**Example 5**

Write the prime factorization for the following number.

\[
144 = 2^4 \times 3^2
\]

The prime factorization of 144 is \(2^4 \times 3^2\).
Review

Write the prime factorization of each number using exponential notation.

1. 56
2. 14
3. 121
4. 84
5. 50
6. 64
7. 72
8. 16
9. 24
10. 300
11. 128
12. 312
13. 525
14. 169
15. 213

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.4.

Resources
3.19 Common Multiples

In this concept, you will learn to identify multiples and find common multiples for pairs of numbers.

Charlie is playing a video game. He has to grab the magic token when the guard turns his back. He notices that the magic token appears every 4 seconds and the guard turns his back every 3 seconds. How can Charlie use this information to get the timing right so he can grab the magic token?

In this concept, you will learn to identify multiples and find common multiples for pairs of numbers.

Finding Common Multiples

A multiple is the product of a quantity and a whole number. Here are some multiples for the quantity of 3, multiplied by different whole numbers.

\[3 \times 1 = 3, \quad 3 \times 2 = 6, \quad 3 \times 3 = 9, \quad 3 \times 4 = 12, \quad 3 \times 5 = 15, \quad 3 \times 6 = 18\]

Listing out these products is the same as listing out multiples.

3, 6, 9, 12, 15, 18...

You can see that this is also the same as counting by threes. The dots at the end mean that these multiples can go on and on and on. Every number has an infinite number of multiples.

List six multiples for 4.

To do this, think of taking the quantity 4 and multiplying it by 1, 2, 3, 4, 5...

\[4 \times 1 = 4, \quad 4 \times 2 = 8, \quad 4 \times 3 = 12, \quad 4 \times 4 = 16, \quad 4 \times 5 = 20\]

Our answer is 4, 8, 12, 16, 20, 24...

A common multiple is a multiple that two or more numbers have in common. List the multiples of the numbers to find the common multiples.

Find common multiples for 3 and 4.

First, write out the first few multiples for the numbers and then identify the multiples the two numbers have in common.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36
4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48
Some of the common multiples of 3 and 4 are 12, 24, and 36.

**Examples**

**Example 1**

Earlier, you were given a problem about Charlie and his video game.

Charlie has to grab the magic token that appears every 4 seconds when the guard turns his back, which is every 3 seconds. Find the common multiples of 3 and 4 to find out when both will occur at the same time.

First, list the first few multiples of 3 and 4.

3, 6, 9, 12, 15, 18, 21, 24, 27
4, 8, 12, 16, 20, 24, 28, 32, 36

Then, identify the common multiples.

3, 6, 9, **12**, 15, 18, 21, **24**, 27
4, 8, **12**, 16, 20, **24**, 28, 32, 36

Some common multiples of 3 and 4 are 12 and 24. Charlie can grab the magic token 12 or 24 seconds after he enters that stage. Note that 12 and 24 are multiples of 12. Charlie will get a chance to grab the magic token every 12 seconds.

**Example 2**

What are common multiples of 3 and 7?

First, write out the first few multiples for the numbers.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30
7, 14, 21, 28, 35, 42, 49, 56, 63, 70

Then, identify the multiples the two numbers have in common.

3, 6, 9, 12, 15, 18, **21**, 24, 27, 30
7, 14, **21**, 28, 35, 42, 49, 56, 63, 70

A common multiple of 3 and 7 is 21.

In this list, the only common multiple between 3 and 7 is 21. You could find more multiples by increasing the length of the list of multiples for 3 and 7 or finding a multiple of the common multiple.

21 × 2 = 42

42 is a multiple of both 3 and 7.

**Example 3**

List eight multiples of 6.

First, multiply 6 by 1 through 8.

6 × 1 = 6, 6 × 2 = 12, 6 × 3 = 18, 6 × 4 = 24, 6 × 5 = 30, 6 × 6 = 36, 6 × 7 = 42, 6 × 8 = 48

The multiples of 6 are 6, 12, 18, 24, 30, 36, 42, and 48.
Example 4

List six multiples of 8.
First, multiply 8 by 1 through 6.

\[
\begin{align*}
8 \times 1 &= 8, \\
8 \times 2 &= 16, \\
8 \times 3 &= 24, \\
8 \times 4 &= 32, \\
8 \times 5 &= 40, \\
8 \times 6 &= 48 \\
\end{align*}
\]

Six multiples of 8 are 8, 16, 24, 32, 40, and 48.

Example 5

What are the first two common multiples of 6 and 8?
First, list the first few multiples of 6 and 8.

6, 12, 18, 24, 30, 36, 42, 48 ...
8, 16, 24, 32, 40, 48, ...

Then, identify the common multiples.

6, 12, 18, 24, 30, 36, 42, 48 ...
8, 16, 24, 32, 40, 48 ...

Two common multiples of 6 and 8 are 24 and 48.

Review

List the first five multiples for each of the following numbers.

1. 3
2. 5
3. 6
4. 7
5. 8

Find two common multiples of each pair of numbers.

6. 3 and 5
7. 2 and 3
8. 3 and 4
9. 2 and 6
10. 3 and 9
11. 5 and 7
12. 4 and 12
13. 5 and 6
14. 10 and 12
15. 5 and 8

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.9.
3.19. Common Multiples

Resources

Compute fluently with multi-digit numbers and find common factors and multiples.

6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.

6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

6.NS.4 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express 36 + 8 as 4 (9 + 2).
Apply and extend previous understandings of numbers to the system of rational numbers.

6.NS.5 Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., -(-3) = 3, and that 0 is its own opposite.

b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

6.NS.7 Understand ordering and absolute value of rational numbers.

a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. For example, interpret -3 > -7 as a statement that -3 is located to the right of -7 on a number line oriented from left to right.

b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. For example, write -3°C > -7°C to express the fact that -3°C is warmer than -7°C.

c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. For example, for an account balance of $-30$ dollars, write $| -30 | = 30$ to describe the size of the debt in dollars.

d. Distinguish comparisons of absolute value from statements about order. For example, recognize that an account balance less than $-30$ dollars represents a debt greater than $30$ dollars.
6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.
4.1 Additive Inverses and Absolute Values

Here you’ll learn how to find the opposite of a number and also its distance from zero on a number line, which is its absolute value.

Suppose that you are creating a budget and that you have expenses of $2500 per month. How much money would you have to bring in each month in order to break even? Would the expenses be thought of as a positive or negative number? What would be the additive inverse of the expenses? What would be the absolute value?

Adding Inverses and Absolute Values

Graphing and Comparing Integers

More specific than the rational numbers are the integers. Integers are whole numbers and their negatives. When comparing integers, you will use the math verbs such as less than, greater than, approximately equal to, and equal to. To graph an integer on a number line, place a dot above the number you want to represent.

Let’s compare the following integers:

2 and -5.

First, we will plot the two numbers on a number line.

![Number Line Diagram]

We can compare integers by noting which is the greatest and which is the least. The greatest number is farthest to the right, and the least is farthest to the left.

In the diagram above, we can see that 2 is farther to the right on the number line than -5, so we say that 2 is greater than -5. We use the symbol > to mean “greater than.”

Therefore, $2 > -5$.

Numbers and Their Opposites

Every real number, including integers, has an opposite, which represents the same distance from zero but in the other direction.
A special situation arises when adding a number to its opposite. The sum is zero. This is summarized in the following property:

- **The Additive Inverse Property**: For any real number \( a \), \( a + (-a) = 0 \)

We see that \(-a\) is the additive inverse, or opposite, of \( a \).

**Now, let’s find the opposite number of the following numbers:**

1. -5

   The opposite number of -5 is 5. Using the Additive Inverse Property: \(-5 + (5) = 0\).

2. 1/2

   The opposite number of 1/2 is -1/2. The Additive Inverse Property shows us that they are opposites: \( \frac{1}{2} + (-\frac{1}{2}) = 0 \).

3. 5.1

   The opposite number of 5.1 is -5.1. The Additive Inverse Property gives: \( 5.1 + (-5.1) = 0 \).

**Absolute Value**

**Absolute value** represents the distance from zero when graphed on a number line. For example, the number 7 is 7 units away from zero. The number -7 is also 7 units away from zero. The absolute value of a number is the distance it is from zero, so the absolute value of 7 and the absolute value of -7 are both 7. A number and its additive inverse are always the same distance from zero, and so they have the same absolute value.

We write the absolute value of -7 like this: \(|-7|\).

We read the expression \(|x|\) like this: “the absolute value of \( x \).”

- Treat absolute value expressions like parentheses. If there is an operation inside the absolute value symbols, evaluate that operation first.
- The absolute value of a number or an expression is always positive or zero. It cannot be negative. With absolute value, we are only interested in how far a number is from zero, not the direction.

**Finally, let’s evaluate the following absolute value expressions:**

1. \(|5 + 4|\)

   \[
   |5 + 4| = |9| = 9
   \]

2. \(3 - |4 - 9|\)
3 – |4 – 9| = 3 – |–5|  
= 3 – 5  
= –2

3. |–5 – 11|

|–5 – 11| = |–16|  
= 16

4. –|7 – 22|

–|7 – 22| = –|–15|  
= –(15)  
= –15

Examples

Example 1

Earlier, you were told that you should create a budget. You have expenses of $2500 per month. How much money would you have to bring in each month in order to break even? Would the expenses be thought of as a positive or negative number? What would be the additive inverse of the expenses? What would be the absolute value?

The word expenses indicates that you are spending $2500 a month. Therefore, the expenses should be thought of a negative number because that is the amount of money that is leaving your bank account. With the expenses, you
have -$2500 in your bank account and to break even, you would need to make enough money to have $0 in your bank account. In other words, you need to find the additive inverse. The additive inverse of -2500 is 2500. So you would need to make $2500 to break even each month. The absolute value of both the expenses and the amount you need to earn is 2500.

**Example 2**

What is the opposite of \(x - 1\)?

The opposite of \(x - 1\) is \(-(x - 1)\). We can use the Additive Inverse Property to prove it:

Since \((-x - 1) = -x - 1\), we can see that

\[(x - 1) + (- (x - 1)) = (x - 1) + (-x - 1) = 0.\]

**Example 3**

Evaluate the following:

\[|3 - 4| - 2\]

\[|3 - 4| - 2 = |-1| - 2 = 1 - 2 = -1\]

**Example 4**

Evaluate the following

\[|5 - 7.5| + 3\]

\[|5 - 7.5| + 3 = |−2.5| + 3 = 2.5 + 3 = 5.5\]

**Review**

1. Define absolute value.
2. Give an example of a real number that is not an integer.
3. The tick-marks on the number line represent evenly spaced integers. Find the values of \(a, b, c, d,\) and \(e\).

In 4-9, find the opposite of each of the following.

4. 1.001
5. -9.345
6. (16 - 45)
7. (5 - 11)
8. \((x + y)\)
9. \((x - y)\)

In 10-19, simplify.

10. \(|−98.4|\)
11. \(|123.567|\)
In 20-25, compare the two real numbers.

20. 8 and 7.99999
21. -4.25 and $-\frac{17}{4}$
22. 65 and -1
23. 10 units left of zero and 9 units right of zero
24. A frog is sitting perfectly on top of number 7 on a number line. The frog jumps randomly to the left or right, but always jumps a distance of exactly 2. Describe the set of numbers that the frog may land on, and list all the possibilities for the frog’s position after exactly 5 jumps.

25. Will a real number always have an additive inverse? Explain your reasoning.

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 2.2.
Here you will review the number sets that make up the real number system. In addition, you will graph inequalities on a real number line. Can you describe the number 13? Can you say what number sets the number 13 belongs to?

**Real Number Line Graphs**

All of the numbers you have learned about so far in math belong to the real number system. Positives, negatives, fractions, and decimals are all part of the real number system. The diagram below shows how all of the numbers in the real number system are grouped.

In practice, there may be more than one symbol or combination of symbols used to describe a particular set of numbers. For example, the set of all irrational numbers may be defined as $\mathbb{Q}$, or $\mathbb{I}$, or even $\mathbb{R} - \mathbb{Q}$ (Real Numbers minus Rational Numbers). The most important thing is to be consistent, and to state the definition of the symbol(s) you use.

Any number in the real number system can be plotted on a real number line. You can also graph inequalities on a real number line. In order to graph inequalities, make sure you know the following symbols:

- The symbol $>$ means “is greater than.”
- The symbol $<$ means “is less than.”
- The symbol $\geq$ means “is greater than or equal to.”
- The symbol $\leq$ means “is less than or equal to.”

The inequality symbol indicates the type of dot that is placed on the beginning point and the number set indicates whether an arrow is drawn on the number line or if points are used.

**Let’s practice using a number line:**

1. Represent $x > 4$ where $x$ is an integer, on a number line.
2. Represent this inequality statement on a number line \( \{x \geq -2 | x \in R\} \).

The statement can be read as "\( x \) is greater than or equal to -2, such that \( x \) belongs to or is a member of the real numbers." In other words, represent all real numbers greater than or equal to -2.

The inequality symbol says that \( x \) is greater than or equal to -2. This means that -2 is included in the graph. A solid dot is placed on -2 and on all numbers to the right of -2. The line is on the number line to indicate that all real numbers greater than -2 are also included in the graph.

3. Represent this inequality statement, also known as set notation, on a number line \( \{x | 2 < x \leq 7, x \in N\} \).

This inequality statement can be read as \( x \) such that \( x \) is greater than 2 and less than or equal to 7 and \( x \) belongs to the natural numbers. In other words, all natural numbers greater than 2 and less than or equal to 7.

The inequality statement that was to be represented on the number line had to include the natural numbers greater than 2 and less than or equal to 7. These are the only numbers to be graphed. There is no arrow on the number line.

**Examples**

**Example 1**

Earlier, you were asked to describe the number 13 and to identify what number set(s) 13 belongs to.

The number 13 is a natural number because it is in the set \( N = \{1, 2, 3, 4 \ldots\} \).

The number 13 is a whole number because it is in the set \( W = \{0, 1, 2, 3 \ldots\} \).

The number 13 is an integer because it is in the set \( Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \).

The number 13 is a rational number because it is in the set \( Q = \{\frac{a}{b}, b \neq 0\} \).

The number 13 belongs to the real number system.
4.2. Real Number Line Graphs

Example 2

Check the set(s) to which each number belongs. The number may belong to more than one set.

<table>
<thead>
<tr>
<th>Number</th>
<th>N</th>
<th>W</th>
<th>Z</th>
<th>Q</th>
<th>( \overline{Q} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>-47</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>1.48</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>( \sqrt{7} )</td>
<td>X</td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>( \pi )</td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

Review the definitions for each set of numbers.

Example 3

Graph \( \{x | -3 \leq x \leq 8, x \in R \} \) on a number line.

The set notation means to graph all real numbers between -3 and +8. The line joining the solid dots represents the fact that the set belongs to the real number system.

Example 4

Use set notation to describe the set shown on the number line.

The closed dot means that -3 is included in the answer. The remaining dots are to the right of -3. The open dot means that 2 is not included in the answer. This means that the numbers are all less than 2. Graphing on a number
line is done from smallest to greatest or from left to right. There is no line joining the dots so the variable does not belong to the set of real numbers. However, negative whole numbers, zero and positive whole numbers make up the integers. The set notation that is represented on the number line is \( \{x| -3 \leq x < 2, x \in \mathbb{Z}\} \).

**Review**

Describe each set notation in words.

1. \( \{x|x > 8, x \in \mathbb{R}\} \)
2. \( \{x|x \leq -3, x \in \mathbb{Z}\} \)
3. \( \{x|-4 \leq x \leq 6, x \in \mathbb{R}\} \)
4. \( \{x|5 \leq x \leq 11, x \in \mathbb{W}\} \)
5. \( \{x|x \geq 6, x \in \mathbb{N}\} \)

Represent each graph using set notation

6. [Graph Image]

7. [Graph Image]

8. [Graph Image]

9. [Graph Image]

10. [Graph Image]
For each of the following situations, use set notations to represent the limits.

11. To ride the new tilt-a-whirl at the fairgrounds, a child can be no taller than 4.5 feet.
12. A dance is being held at the community hall to raise money for breast cancer. The dance is only for those people 19 years of age or older.
13. A sled driver in the Alaska Speed Quest must start the race with no less than 10 dogs and no more than 16 dogs.
14. The residents of a small community are planning a skating party at the local lake. In order for the event to take place, the outdoor temperature needs to be above \(-6^\circ C\) and not above \(-1^\circ C\).
15. Juanita and Hans are planning their wedding supper at a local venue. To book the facility, they must guarantee that at least 100 people will have supper but no more than 225 people will eat.

Represent the following set notations on a number line.

16. \(\{x | x > 6, x \in N\}\)
17. \(\{x | x \leq 8, x \in R\}\)
18. \(\{x | -3 \leq x < 6, x \in Z\}\)

**Review (Answers)**

To see the Review answers, open this [PDF file](#) and look for section 1.14.
4.3 Ordered Pairs in Four Quadrants

In this concept, you will learn to graph ordered pairs in four quadrants.

Liam has a crush on Lila. She’s a little strange—she has an old school phone booth on her key chain and he suspects that the strange characters that she doodles on her notebook during history class might actually be elvish runes—but he likes her anyway. When Liam asks her to the Spring Fling, it doesn’t really surprise him when she responds with a list of coordinates: (-3,5), (3,5), (0,0), (0,-5). Still, he’s left wondering whether that’s a yes or a no, and he doesn’t want to lose face by asking her. How can Liam figure out what this means?

In this concept, you will learn how to graph coordinates in four quadrants.

Graphing Coordinates in Four Quadrants

A coordinate grid is a grid in which points are graphed. It usually has two or more intersecting lines which divide a plane into quadrants, and in which ordered pairs, or coordinates, are defined. The coordinate grid below has one quadrant, or section, to it.

The origin is the place where the two lines intersect. Its coordinates are defined as (0,0).

The x-axis is the line running from left to right that has the numbers defined on it and is usually labeled with an "x". The x-coordinate of an ordered pair is found with relation to it. All the points located on the x-axis have a y-coordinate of 0.

The y-axis is the central line that runs up-down and is labeled with a "y". Y-coordinates are plotted in reference to this axis. Again, all the x-coordinates of points located on the y-axis are 0.

An ordered pair is a list of two numbers in parenthesis, separated by a comma like this: (5,-3). It tells where a point is located on the coordinate plane. The first number is the x-coordinate. It tells you where to go on the x-axis. If it is positive, you go to the right. If it is negative, you go to the left. The second number is the y-coordinate. It tells you where to go on the y-axis. If it is positive, you go up. If it is negative, you go down.

Here is an example.

Plot (3, 5) on the coordinate grid then label it point A.
First, look at the x-coordinate.
In this case, it is 3. So you go 3 to the right.
Next, look at the y-coordinate.
In this case, it is 5. So you go up 5.
Then, you draw a dot at that place.

Most coordinate grids have four quadrants. They look like this:

Here is an example of a point graphed in a four-quadrant coordinate grid. Graph the point (-4, 3) and name it point P.
First, look at the x-coordinate.
In this case, it is -4. "-" means move to the left, so you go 4 to the left.
Next, look at the y-coordinate.
In this case, it is 3. It is positive, so that means it is up. Go up 3 units.
The result looks like this:

![Graph showing the movement from the origin to the point P]

**Examples**

**Example 1**

Earlier, you were given a problem about Liam and his girl troubles.
Lila gave him a list of ordered pairs—(-3,5), (3,5), (0,0), (0,-5)—when he asked her out. Did she say yes or no?
First, Liam takes some graph paper and jots a coordinate plane.
Next, he sees that one of the points is at the origin, so he draws a dot there.
Then, he moves to the left 3 and up 5 and makes a point there.
Then, he sees that there is another point at 5 on the y-axis, so he goes ahead and moves over to +3 on the x-axis and makes a dot there, opposite the other point. At this point, he has a V and he thinks it could go either way.
Finally, he goes down 5 on the y-axis. There are no other points, so that looks like his answer. Sweet! Now, he just needs to figure out what he’s going to wear.

**Example 2**

Write the coordinates for the following point.
Begin at the origin. Move five units to the right and three units down.
First, figure out the x-coordinate of the point.
The x-axis is the left-right axis. Right is the positive direction. So, the x-coordinate of the point is 5.
Next, figure out the y-coordinate.
The y-axis is the up-down axis. Down is the negative direction. So, the y-coordinate is -3.
Then, write the ordered pair of the point.
The answer is (5, -3).

**For the following examples, identify each ordered pair on the coordinate grid given.**

**Example 3**

Point A  
First, locate point A. 
Next, determine the x-coordinate. 
It is at +1 on the x-axis. 
Then, determine the y-coordinate. 
It is at +1 on the y-axis. 
Finally, write the ordered pair. 
The answer is (1,1)

**Example 4**

Point B  
First, locate point B. 
Next, determine the x-coordinate. 
It is at -3 on the x-axis. 
Then, determine the y-coordinate. 
It is at -1 on the y-axis. 
Finally, write the ordered pair. 
The answer is (-3,-1)

**Example 5**

Point C  
First, locate point C. 
Next, determine the x-coordinate. 
It is on the y-axis without going left or right. That means the x-coordinate is 0. 
Then, determine the y-coordinate. 
It is at +4 on the y-axis. 
Finally, write the ordered pair. 
The answer is (0,4)
Example 6

Point D
First, locate point D.
Next, determine the x-coordinate.
It is at +2 on the x-axis.
Then, determine the y-coordinate.
It is at -3 on the y-axis.
Finally, write the ordered pair.
The answer is (2,-3)

Review

Identify the coordinates of each of the points plotted on the coordinate grid.

1. $A$
2. $B$
3. $C$
4. $D$
5. $E$
6. $F$
7. $G$
8. $H$
9. $I$
10. $J$

Answer the following questions.

11. What is the center point called?
12. What are it’s coordinates?
13. If you move to the right of the origin, are the values positive or negative?
14. What is the horizontal line called?
15. What is the vertical line called?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 11.14.
4.4 Absolute Value

Here you’ll review the meaning of absolute value and learn how to find the distance between two numbers on a number line.

Suppose two people stood back-to-back and then walked in opposite directions. The first person walked 2 miles, while the second person walked 5 miles. How far apart would they be?

Absolute Value

The absolute value of a number is the distance from zero on a number line. The numbers 4 and -4 are each four units away from zero on a number line. So, \(|4| = 4\) and \(|-4| = 4\).

Below is a more formal definition of absolute value.

For any real number \(x\),

\[
|x| = x \text{ for all } x \geq 0 \\
|x| = -x (\text{read the opposite of } x) \text{ for all } x < 0
\]

The second part of this definition states that the absolute value of a negative number is its opposite (a positive number).

Let’s evaluate the following absolute value:

\(|-120|\)

The absolute value of a negative number is its inverse, or opposite. Therefore, \(|-120| = -(-120) = 120\).

Distance on the Number Line

Because the absolute value is always positive, it can be used to find the distance between two values on a number line.

The distance between two values \(x\) and \(y\) on a number line is found by:

\[
distance = |x - y| \text{ or } |y - x|
\]

Let’s use absolute value to find the following values:

1. The distance between -5 and 8.

   Use the definition of distance. Let \(x = -5\) and \(y = 8\).

   \[
distance = |-5 - 8| = |-13|
\]
The absolute value of -13 is 13, so -5 and 8 are 13 units apart. Check on the graph below that the length of the line between the points -5 and 8 is 13 units long:

2. The distance between -12 and 3.

Use the definition of distance. Let \( x = 3 \) and \( y = -12 \).

\[
distance = |3 - (-12)| = |3 + 12| = |15| = 15
\]

The absolute value of 15 is 15, so 3 and -12 are 15 units apart.

---

Examples

Example 1

Earlier, you were told that two people stood back-to-back and walked in opposite directions. The first person walked 2 miles and the second person walked 5 miles. How far apart would they be?

This is a distance problem. Assume that the point that the people start walking from is 0 on the number line and that the person walking 2 miles was walking to the left or in the negative direction. Then the person who walked 5 miles is walking to the right or the positive direction. Therefore, you want to find the distance between -2 and 5. You can use the distance formula shown in this section.

\[
distance = |5 - (-2)| = |5 + 2| = |7| = 7
\]

The two people are 7 miles apart.
Example 2

Find the distance between 8 and -1.

We use the distance formula. Let \( x = 8 \) and \( y = -1 \).

\[
distance = |8 - (-1)| = |8 + 1| = |9| = 9
\]

Notice that if we let \( x = -1 \) and \( y = 8 \), we get:

\[
distance = |-1 - 8| = |-9| = 9
\]

So, it does not matter which number we pick for \( x \) and \( y \). We will get the same answer.

Review

Evaluate the absolute value.

1. \(|250|\)
2. \(|-12|\)
3. \(|-\frac{2}{5}|\)
4. \(|\frac{1}{11}|\)

Find the distance between the points.

5. 12 and -11
6. 5 and 22
7. -9 and -18
8. -2 and 3
9. \(\frac{2}{3}\) and -11
10. -10.5 and -9.75
11. 36 and 14

Mixed Review

12. Solve: \(6t - 14 < 2t + 7\).
13. The speed limit of a semi-truck on the highway is between 45 mph and 65 mph.
   a. Write this situation as a compound inequality
   b. Graph the solutions on a number line.
14. Lloyd can only afford transportation costs of less than $276 per month. His monthly car payment is $181 and he sets aside $25 per month for oil changes and other maintenance costs. How much can he afford for gas?
15. Simplify \(\sqrt{12} \times \sqrt{3}\).
16. A hush puppy recipe calls for 3.4 ounces of flour for one batch of 8 hush puppies. You need to make 56 hush puppies. How much flour do you need?
17. What is the additive inverse of 124?
18. What is the multiplicative inverse of 14?
4.4. Absolute Value

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.7.
In this concept, you will learn to write and graph inequalities.

A dance is being held at the High School and is only open for teenagers who are 18 or younger. Arthur must make posters to advertise the dance in all of the Math classrooms in the school. He wants to make sure the age restriction is included on the poster. How can Arthur mathematically display the age restriction on his posters?

In this concept, you will learn to write and graph inequalities.

**Inequalities**

An equation is a statement of equality between two quantities. A statement that one quantity may or may not be equal to another quantity is called an inequality. The statement expressed by an equation is modeled using an equal sign, while the statement expressed by an inequality is modeled using the following symbols:

- The symbol $<$ means “is less than.”
- The symbol $>$ means “is greater than.”
- The symbol $\leq$ means “is less than or equal to.”
- The symbol $\geq$ means “is greater than or equal to.”

The line below the inequality symbols for less than and greater than indicates the ‘equal to’ part of the meanings of $\leq$ and $\geq$. If you have difficulty remembering which symbol means less than and which one means greater than, remember “l-e-s-s” points “l-e-f-t,” or that the smaller side of the $<$ sign (the point) is the smaller number.

Let’s take a closer look at inequalities.
2 \ > x

This inequality means “2 is greater than x.” There are numerous values for the variable ‘x’ that would make this statement true. Some of the numbers that could be substituted in for the variable can be listed as shown here:{1, 0, −1, …} These values belong to the integers which are a subset of the real numbers.

Another way to show these same numbers is by using set notation. Set notation is a mathematical statement that shows an equation or inequality and the set of numbers to which the variable belongs. For the above inequality the values for the variable ‘x’ can be represented using set notation as shown below:

\{x \ : \ 2 > x, \ x \in \mathbb{I}\}

Let’s look at another example.

5 \ \leq y

This inequality means “5 is less than or equal to y.” OR “y is greater than or equal to 5.” Some of the numbers that could be substituted in for the variable ‘y’ are \{5, 5.5, 6, 7\frac{1}{4}, …\} These values belong to the set of rational numbers, which make up a subset of the real numbers. To represent the values for the variable using set notation you could write \{y \ : \ 5 \leq y, \ y \in \mathbb{Q}\}. The letter \mathbb{Q} represents the rational numbers. The letter \mathbb{R} could also be used here to represent the real numbers.

Inequalities can also be graphed on a number line. The graph visually displays the set of numbers for the variable that would create a true statement. Remember there are four symbols that can be used to write an inequality. Each of these can be displayed on the number line graph of the inequality. Here are some hints to use when graphing an inequality on a number line:

- If the inequality is written with either a “less than” (<) or a “greater than” (>) symbol, then the starting number of your graph is not included in the solution of the inequality. This is represented by placing an open circle (◦) on that value on the number line.
- If the inequality is written with either a “less than or equal to” (≤) or a “greater than or equal to” (≥) symbol, then the starting number of your graph is included in the solution of the inequality. This is represented by placing a closed circle (●) on that value on the number line.
- To represent “less than” draw an arrow on the number line pointing left. To represent “greater than” draw an arrow on the number line pointing right.
- If the values of the variable belong to the rational numbers or to the real numbers, draw a line joining the plotted values. If the values belong to the natural numbers, whole numbers, or integers do not join the plotted points.

Write an inequality to represent the set of all possible values of ‘n ’ if n is less than two. Then, graph the inequality on a number line.

First, write an inequality to represent the given information.
The inequality is $n < 2$. This inequality must be graphed on a number line.

First, draw a number line numbered from -5 to 5.

Next, the number 2 is not included in the solution since the values for the variable will be all numbers less than 2. Draw an open circle on the number line at '2'.

Then, draw a direction line to the left from 2 to indicate “less than” and the fact that all numbers less than 2 are included in the solution. This means that $n \in \mathbb{R}$.

---

**Examples**

**Example 1**

Earlier, you were given a problem about Arthur and his dance posters. He wants to make sure that all students know the age requirements for attending the dance. How can Arthur do this mathematically?

First, create an eye-catching poster advertising the dance.

Next, write a verbal model to represent the age restriction. "You must be between the ages of 13 and 18."

Verbal Model: To be admitted to the dance your age must be “greater than or equal to 13” and “less than or equal to 18.”

Next, name the variable. Let 'a' represent the age of the students.

Next, use set notation to write an inequality to represent the verbal model.

$$\{a : 13 \leq a \leq 18, \ a \in \mathbb{R}\}$$

**Example 2**

Write an inequality to represent the set of all possible values of 'n' if $n$ is greater than or equal to negative four. Then, graph the inequality on a number line.

First, write an inequality to represent the given information.

$$n \quad \text{is greater than or equal to negative four}$$

$$n \geq -4$$
The inequality is $n \geq -4$. This inequality must be graphed on a number line. 
First, draw a number line numbered from -5 to 5.
Next, the number -4 is included in the solution since the values for the variable will be all numbers greater than or equal to -4. Draw a closed circle on the number line at '-4'.
Then, draw a direction line to the right from -4 to indicate “greater than” and the fact that all numbers greater than -4 are included in the solution. This means that $n \in \mathbb{R}$.

**Example 3**

Draw a number line graph to represent the following inequality expressed in set notation.

\[
\{ x : 3 \leq x < 2, x \in \mathbb{I} \}
\]

First, write in words what the set notation represents. Remember to read it from the middle beginning with the variable.
The set notation represents all integers “greater than or equal to -3” and “less than” 2.
Next, draw a number line and place a closed circle on -3 since it is included in the solution and an open circle on 2 since it is not included in the solution.
Then, since the integers between these values are included in the solution place closed circles on {-2, -1, 0, and 1}. Do not join the circles with a line since ‘$x$’ belongs to the Integers.

**Example 4**

Write an inequality to represent the following statement:
The quantities less than or equal to four.
First, name the variable. Let ‘$x$’ represent the quantities in the set.
Next, write the symbol that means “less than or equal to.” \( \leq \)
Next, write the number. 4
Then, write the inequality.

\[ x \leq 4 \]

The answer is $x \leq 4$. 

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Example 5

Represent the following using set notation:
All real numbers greater than -2 and less than or equal to 11.
First, name the variable. Let \( n \) represent all real numbers in the set.
Next, begin the format for writing set notation. \( \{ n : \} \)
Next, write the smaller number given in the statement.
\[
\{ n : -2 \}
\]
Next, insert the symbol that means “less than.”
\[
\{ n : -2 < \}
\]
Next, write the variable.
\[
\{ n : -2 < n \}
\]
Next, write the symbol that means “less than or equal to.”
\[
\{ n : -2 < n \leq \}
\]
Next, write the larger number given in the statement.
\[
\{ n : -2 < n \leq 11, \}
\]
Then, write the set of numbers to which the variable belongs.
\[
\{ n : -2 < n \leq 11, n \in \mathbb{R} \}
\]
The answer is \( \{ n : -2 < n \leq 11, n \in \mathbb{R} \} \).

Review

Write a solution set for each inequality. Include at least three values in your solution set.
1. \( x < 13 \)
2. \( y > 5 \)
3. \( x < 2 \)
4.5. Write and Graph Inequalities

4. \( y > -3 \)
5. \( a > 12 \)
6. \( x \leq 4 \)
7. \( y \geq 3 \)
8. \( b \geq -3 \)
9. \( a \leq -5 \)
10. \( b \geq 11 \)

Write an inequality to describe each situation.

11. A number is less than or equal to -8.
12. A number is greater than 50.
13. A number is less than -4.
14. A number is greater than -12.
15. A number is greater than or equal to 11.

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 3.12.
4.6 Rational Numbers in Applications

Here you’ll apply the properties of addition and subtraction to solve real-world problems involving rational numbers.

Rational Numbers in Applications

Let’s use the skills we learned in the last concept to solve some real-world problems.

Real-World Application: School Trip

Peter is hoping to travel on a school trip to Europe. The ticket costs $2400. Peter has several relatives who have pledged to help him with the ticket cost. His parents have told him that they will cover half the cost. His grandma Zenoviea will pay one sixth, and his grandparents in Florida will send him one fourth of the cost. What fraction of the cost can Peter count on his relatives to provide?

The first thing we need to do is extract the relevant information. Peter’s parents will provide \( \frac{1}{2} \) the cost; his grandma Zenovica will provide \( \frac{1}{6} \); and his grandparents in Florida \( \frac{1}{4} \). We need to find the sum of those numbers, or \( \frac{1}{2} + \frac{1}{6} + \frac{1}{4} \).

To determine the sum, we first need to find the LCD. The LCM of 2, 6 and 4 is 12, so that’s our LCD. Now we can find equivalent fractions:

\[
\begin{align*}
\frac{1}{2} &= \frac{6 \cdot 1}{6 \cdot 2} = \frac{6}{12} \\
\frac{1}{6} &= \frac{2 \cdot 1}{2 \cdot 6} = \frac{2}{12} \\
\frac{1}{4} &= \frac{3 \cdot 1}{3 \cdot 4} = \frac{3}{12}
\end{align*}
\]

Putting them all together: \( \frac{6}{12} + \frac{2}{12} + \frac{3}{12} = \frac{11}{12} \).

Peter will get \( \frac{11}{12} \) the cost of the trip, or $2200 out of $2400, from his family.

Real-World Application: Property Management

A property management firm is buying parcels of land in order to build a small community of condominiums. It has just bought three adjacent plots of land. The first is four-fifths of an acre, the second is five-twelfths of an acre, and
the third is nineteen-twentieths of an acre. The firm knows that it must allow one-sixth of an acre for utilities and a small access road. How much of the remaining land is available for development?

The first thing we need to do is extract the relevant information. The plots of land measure \( \frac{4}{5} \), \( \frac{5}{12} \), and \( \frac{19}{20} \) acres, and the firm can use all of that land except for \( \frac{1}{6} \) of an acre. The total amount of land the firm can use is therefore \( \frac{4}{5} + \frac{5}{12} + \frac{19}{20} - \frac{1}{6} \) acres.

We can add and subtract multiple fractions at once just by finding a common denominator for all of them. The factors of 5, 9, 20, and 6 are as follows:

\[
\begin{align*}
5 & = 1 \cdot 5 \\
12 & = 2 \cdot 2 \cdot 3 \\
20 & = 2 \cdot 2 \cdot 5 \\
6 & = 2 \cdot 3
\end{align*}
\]

We need a 5, two 2’s, and a 3 in our LCD. \( 2 \cdot 2 \cdot 3 \cdot 5 = 60 \), so that’s our common denominator. Now to convert the fractions:

\[
\begin{align*}
\frac{4}{5} &= \frac{12 \cdot 4}{60} = \frac{48}{60} \\
\frac{5}{12} &= \frac{5 \cdot 5}{60} = \frac{25}{60} \\
\frac{5}{20} &= \frac{5 \cdot 20}{60} = \frac{57}{60} \\
\frac{1}{6} &= \frac{10 \cdot 1}{60} = \frac{10}{60}
\end{align*}
\]

We can rewrite our sum as \( \frac{48}{60} + \frac{25}{60} + \frac{57}{60} - \frac{10}{60} = \frac{48 + 25 + 57 - 10}{60} = \frac{120}{60} \).

Next, we need to reduce this fraction. We can see immediately that the numerator is twice the denominator, so this fraction reduces to \( \frac{2}{1} \) or simply 2. One is sometimes called the invisible denominator, because every whole number can be thought of as a rational number whose denominator is one.

The property firm has two acres available for development.

**Evaluate Change Using a Variable Expression**

When we write algebraic expressions to represent a real quantity, the difference between two values is the change in that quantity.

![Light Bulb](Light_Bulb.png) (dist) ![Detector](Detector.png)

*The intensity of light hitting a detector when it is held a certain distance from a bulb is given by this equation:*

\[
\text{Intensity} = \frac{3}{d^2}
\]
where \( d \) is the distance measured in meters, and intensity is measured in lumens. Calculate the change in intensity when the detector is moved from two meters to three meters away.

We first find the values of the intensity at distances of two and three meters.

\[
\text{Intensity (2)} = \frac{3}{(2)^2} = \frac{3}{4} \\
\text{Intensity (3)} = \frac{3}{(3)^2} = \frac{3}{9} = \frac{1}{3}
\]

The difference in the two values will give the change in the intensity. We move from two meters to three meters away.

\[
\text{Change} = \text{Intensity (3)} - \text{Intensity (2)} = \frac{1}{3} - \frac{3}{4}
\]

To find the answer, we will need to write these fractions over a common denominator. The LCM of 3 and 4 is 12, so we need to rewrite each fraction with a denominator of 12:

\[
\frac{1}{3} = \frac{4 \cdot 1}{4 \cdot 3} = \frac{4}{12} \\
\frac{3}{4} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}
\]

So we can rewrite our equation as \( \frac{4}{12} - \frac{9}{12} = -\frac{5}{12} \). The negative value means that the intensity decreases as we move from 2 to 3 meters away.

When moving the detector from two meters to three meters, the intensity falls by \( \frac{5}{12} \) lumens.

Guided Practice

Example 1

Elsa baked a small cake for her family. First her sister ate one quarter and her mom ate one third. How much was left for Elsa?

The whole cake is represented by 1. To solve this problem, we subtract the fraction that each person ate.

\[1 - \frac{1}{4} - \frac{1}{3} \]

To complete this problem, we must give the terms common denominators. Since the denominators do not share any factors, we simply multiply them together: \( 4 \cdot 3 = 12 \).
4.6. Rational Numbers in Applications

\[
1 - \frac{1}{4} - \frac{1}{3} \quad \text{Start with the original expression.}
\]

\[
= 1 \cdot \frac{12}{12} - \frac{1}{4} \cdot \frac{3}{3} - \frac{1}{3} \cdot \frac{4}{4} \quad \text{Give each term a common denominator.}
\]

\[
= \frac{12}{12} - \frac{3}{12} - \frac{4}{12} \quad \text{Simplify.}
\]

\[
= \frac{12 - 3 - 4}{12} = \frac{5}{12}
\]

There is \(\frac{5}{12}\) of the original cake left for Elsa.

**Review**

Which property of addition does each situation involve?

1. Whichever order your groceries are scanned at the store, the total will be the same.
2. However many shovel-loads it takes to move 1 ton of gravel, the number of rocks moved is the same.
3. If Julia has no money, then Mark and Julia together have just as much money as Mark by himself has.

In 4-7, practice your addition and subtraction skills.

4. \(\frac{7}{12} + \frac{2}{9}\)
5. \(\frac{5}{11} + \frac{2}{7}\)
6. \(\frac{17}{9} - \frac{18}{9}\)
7. \(\frac{2}{3} - \frac{1}{4}\)

8. Ilana buys two identically sized cakes for a party. She cuts the chocolate cake into 24 pieces and the vanilla cake into 20 pieces, and lets the guests serve themselves. Martin takes three pieces of chocolate cake and one of vanilla, and Sheena takes one piece of chocolate and two of vanilla. Which of them gets more cake?

9. Nadia, Peter and Ian are pooling their money to buy a gallon of ice cream. Nadia is the oldest and gets the greatest allowance. She contributes half of the cost. Ian is next oldest and contributes one third of the cost. Peter, the youngest, gets the smallest allowance and contributes one fourth of the cost. They figure that this will be enough money. When they get to the check-out, they realize that they forgot about sales tax and worry there will not be enough money. Amazingly, they have exactly the right amount of money. What fraction of the cost of the ice cream was added as tax?

10. The time taken to commute from San Diego to Los Angeles is given by the equation \(\text{time} = \frac{120}{\text{speed}}\) where \(\text{time}\) is measured in hours and \(\text{speed}\) is measured in miles per hour (mph). Calculate the change in time that a rush hour commuter would see when switching from traveling by bus to traveling by train, if the bus averages 40 mph and the train averages 90 mph.

**Review (Answers)**

To view the Review answers, open this PDF file and look for section 2.4.
4.7 Mixed Numbers as Improper Fractions

In this concept, you will learn how to rewrite mixed numbers as an improper fraction.

Casey ordered eight pizzas for the drama club to enjoy. Each pizza had ten slices. At the end of the pizza party, there were two whole pizzas and two slices left. How many slices weren’t eaten? How can you express this as an improper fraction of pizza slices?

In this concept, you will learn how to rewrite mixed numbers as an improper fraction.

Writing Mixed Numbers as Improper Fractions

A **mixed number** is a number that has both wholes and parts in it. Here is a mixed number.

\[ 5 \frac{1}{4} \]

There are five whole items and one-fourth of a whole. The opposite of a mixed number is an improper fraction. An **improper fraction** is a fraction that has a larger numerator than the denominator. Here is an improper fraction.

\[ \frac{12}{5} \]

The denominator tells you how many parts the whole has been divided into. This whole has been divided into 5 parts. The numerator tells you the number of parts. In this case, there are twelve parts. There are more parts than there are in 1 whole.
To write a mixed number as an improper fraction, write the mixed number as a fraction in terms of parts instead of in terms of wholes and parts. Remember that a whole number can also be written as a fraction. The numerator is equal to the denominator.

\[
1 = \frac{5}{5}
\]

Change \(2\frac{1}{3}\) to an improper fraction.
First, multiply the whole number by the denominator to convert the whole to a fraction and add the numerator. This will give you the new numerator.

\[
2 \times 3 + 1 = 7
\]

Then, put the sum over the denominator. The denominator is 3.

\[
2 \frac{1}{3} = \frac{7}{3}
\]

The mixed number \(2\frac{1}{3}\) is also written as \(\frac{7}{3}\).

Change the following mixed numbers to improper fractions.

**Examples**

**Example 1**

Earlier, you were given a problem about Casey and the pizzas.
Convert 2 pizzas and 2 slices to an improper fraction to find the number of uneaten slices of pizza.
Two whole pizzas and two slices = \(2 \frac{2}{10}\)

First, multiply the whole number by the denominator and add the numerator.

\[
2 \times 10 + 2 = 22
\]

Then, put the sum over the denominator. The denominator is 10.

\[
2 \frac{2}{10} = \frac{22}{10}
\]

There were a total of 22 slices of pizza left uneaten.

**Example 2**

Express \(4\frac{7}{8}\) as an improper fraction.
First, multiply the whole number by the denominator and add the numerator.

\[
4 \times 8 + 7 = 39
\]

Then, put the sum over the denominator. The denominator is 8.
$$\frac{47}{8} = \frac{39}{8}$$

The mixed number $4\frac{7}{8}$ is expressed as $\frac{39}{8}$.

**Example 3**

$3\frac{1}{3}$

First, multiply the whole number by the denominator and add the numerator.

$$3 \times 3 + 1 = 10$$

Then, put the sum over the denominator. The denominator is 3.

$$3\frac{1}{3} = \frac{10}{3}$$

The mixed number $3\frac{1}{3}$ is expressed as $\frac{10}{3}$.

**Example 4**

$5\frac{2}{3}$

First, multiply the whole number by the denominator and add the numerator.

$$3 \times 5 + 2 = 17$$

Then, put the sum over the denominator. The denominator is 3.

$$3\frac{2}{3} = \frac{17}{3}$$

The mixed number $5\frac{2}{3}$ is expressed as $\frac{17}{3}$.

**Example 5**

$6\frac{1}{8}$

First, multiply the whole number by the denominator and add the numerator.

$$6 \times 8 + 1 = 49$$

Then, put the sum over the denominator. The denominator is 3.

$$6\frac{1}{8} = \frac{49}{8}$$

The mixed number $6\frac{1}{8}$ is expressed as $\frac{49}{8}$.

**Review**

Write each mixed number as an improper fraction.

1. $2\frac{1}{7}$
2. $3\frac{1}{4}$
3. $5\frac{1}{4}$
4. $4\frac{2}{3}$
5. $6\frac{1}{3}$
6. $6\frac{1}{5}$
4.7. Mixed Numbers as Improper Fractions

7. $7 \frac{1}{8}$
8. $8 \frac{3}{4}$
9. $7 \frac{2}{3}$
10. $8 \frac{5}{6}$
11. $8 \frac{1}{3}$
12. $9 \frac{1}{5}$
13. $6 \frac{7}{8}$
14. $9 \frac{3}{4}$
15. $5 \frac{3}{4}$
16. $16 \frac{1}{4}$

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 5.14.

**Resources**

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/166990
4.8 Improper Fractions as Mixed Numbers

In this concept, you will learn to write improper fractions as mixed numbers.

The school 6th grade class had a bake sale. Missy brought 48 muffins to sell. At the end of the day, there were still 15 muffins left. How many dozen muffins were left? Write the amount as a mixed number.

In this concept, you will learn to write improper fractions as mixed numbers.

Writing Improper Fractions as Mixed Numbers

An improper fraction is a fraction where the numerator is larger than the denominator. An improper fraction can be written as a mixed number. A mixed number is composed of a whole number and a fraction.

To change an improper fraction to a mixed number, divide the numerator by the denominator. This will tell you the number of wholes. If there is a remainder, it is the fraction part of a mixed number.

Here is an improper fraction.

\[
\frac{18}{4}
\]

There are 18 parts and the whole has only been divided into 4 parts. Remember that when the numerator is larger than the denominator, there is more than one whole.

\[
1 \text{ whole } = \frac{4}{4}
\]

Convert \(\frac{18}{4}\) to a mixed number.
First, divide the numerator by the denominator.

\[ 18 \div 4 = 4R2 \]

Then, write the quotient as a mixed number with the remainder as a fraction. The remainder is the numerator of the fraction.

\[ \frac{18}{4} = 4\frac{2}{4} \]

Next, look at the fraction. Simplify the fraction if you can. Divide the numerator and denominator by the greatest common factor, 2.

\[ \frac{2}{4} = \frac{1}{2} \]

The improper fraction \( \frac{18}{4} \) is expressed as \( 4\frac{2}{4} \) or \( 4\frac{1}{2} \).

Sometimes, you will have an improper fraction that converts to a whole number and not a mixed number.

\[ \frac{18}{9} \]

Here 18 divided by 9 is 2. There is no remainder, so there is no fraction. This improper fraction converts to a whole number.

The improper fraction \( \frac{18}{9} \) is expressed as 2.

**Examples**

**Example 1**

Earlier, you were given a problem about Missy and her muffins.

Missy had 15 muffins left over from the bake sale and a dozen contains 12 muffins. Convert 15 muffins as a fraction out of 12 to find the number of dozen muffins left.

\[ \text{Muffins left over} = \frac{15}{12} \]

First, divide the numerator by the denominator.

\[ 15 \div 12 = 1R3 \]

Then, write the quotient as a mixed number with the remainder as a fraction.
\[ \frac{15}{12} = 1 \frac{3}{12} \]

Next, look at the fraction. Simplify the fraction if you can. Divide the numerator and denominator by the greatest common factor, 3.

\[ \frac{3}{12} = \frac{1}{4} \]

There were \(1 \frac{1}{4}\) dozen muffins left over.

**Example 2**

Express this improper fraction as a mixed number.

\[ \frac{82}{5} \]

First, divide the numerator by the denominator.

\[ 82 \div 5 = 16R2 \]

Then, write the quotient as a mixed number with the remainder as a fraction. The remainder is the numerator of the fraction.

\[ \frac{82}{5} = 16 \frac{2}{5} \]

The improper fraction \(\frac{82}{5}\) is expressed as \(16 \frac{2}{5}\).

**Example 3**

Express this improper fraction as a mixed number.

\[ \frac{24}{5} \]

First, divide the numerator by the denominator.

\[ 24 \div 5 = 4R4 \]

Then, write the quotient as a mixed number with the remainder as a fraction. The remainder is the numerator of the fraction.

\[ \frac{24}{5} = 4 \frac{4}{5} \]

The improper fraction \(\frac{24}{5}\) is expressed as \(4 \frac{4}{5}\).
Express this improper fraction as a mixed number.

\[
\frac{21}{3}
\]

First, divide the numerator by the denominator.

\[
21 \div 3 = 7
\]

This fraction has no remainder and is not a mixed number.

The improper fraction \(\frac{21}{3}\) is equal to 7.

**Example 5**

Express this improper fraction as a mixed number.

\[
\frac{32}{6}
\]

First, divide the numerator by the denominator.

\[
32 \div 6 = 5R2
\]

Then, write the quotient as a mixed number with the remainder as a fraction. The remainder is the numerator of the fraction.

\[
\frac{32}{6} = 5\frac{2}{6}
\]

Next, look at the fraction. Simplify the fraction if you can. Divide the numerator and denominator by the greatest common factor, 2.

\[
\frac{2}{6} = \frac{1}{3}
\]

The improper fraction \(\frac{32}{6}\) is expressed as \(5\frac{2}{6}\) or \(5\frac{1}{3}\).

**Review**

Convert each improper fraction to a mixed number. Simplify when necessary.

1. \(\frac{22}{3}\)
Review (Answers)

To see the Review answers, open this PDF file and look for section 5.15.

Resources

MEDIA

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/167001
In this concept, you will learn how to compare and order improper fractions and mixed numbers.

Keith and his sister were assigned the task of cleaning up after a party. Keith took all of the leftover tuna sandwiches and his sister took all of the left over ham sandwiches.
Keith has $\frac{15}{2}$ of tuna sandwiches.  
His sister has $6\frac{3}{4}$ of ham sandwiches. Who has more sandwiches?  

In this concept, you will learn how to compare and order improper fractions and mixed numbers.

**Comparing Improper Fractions and Mixed Numbers**

An **improper fraction** is a fraction where the numerator is larger than the denominator.  
A **mixed number** is composed of a whole number and a fraction.  

To compare a mixed number and an improper fraction, first make sure that they are in the same form. Convert the improper fraction to a mixed number or the mixed number to an improper fraction, then compare.

\[
\begin{align*}
6\frac{1}{2} & \quad \frac{15}{4} \\
\end{align*}
\]

Convert $\frac{15}{4}$ into a mixed number. Divide 15 by 4 and write the quotient as a whole number and a fraction.

\[
\frac{15}{4} = 3 \frac{3}{4}
\]

Compare the numbers.

\[
6\frac{1}{2} > 3\frac{3}{4}
\]

$6\frac{1}{2}$ is greater than $\frac{15}{4}$.

If the whole number is the same, compare the fractions. You may have to convert the fractions using the lowest common denominator.

You can order improper fractions and mixed numbers in the same way. Convert them all to the same form and then write them in order.

Order these fractions from least to greatest.

\[
\begin{align*}
4\frac{1}{2}, & \quad 10\frac{4}{5}, \quad 3\frac{7}{9}
\end{align*}
\]

First, change the fractions so that they are all in the same form. Let’s change them all to mixed numbers. Simplify if you can.

\[
\begin{align*}
10 & = \frac{5}{1} = 1\frac{2}{3} \\
4 & = 1\frac{1}{3}
\end{align*}
\]

Now you can write them in order from least to greatest.

\[
\begin{align*}
\frac{4}{3}, & \quad \frac{10}{6}, \quad \frac{4}{2}, \quad \frac{7}{9}
\end{align*}
\]
**Examples**

**Example 1**

Earlier, you were given a problem about Keith and the sandwiches. Keith has $\frac{15}{2}$ tuna sandwiches and his sister has $6\frac{3}{4}$ ham sandwiches. Compare the fractions to see who has more sandwiches.

First, convert the improper fraction to a mixed number.

$$\frac{15}{2} = 7\frac{1}{2}$$

Then, compare the two quantities.

$$7\frac{1}{2} > 6\frac{3}{4}$$

Keith has more sandwiches.

*For the following examples, compare the fractions.*

**Example 2**

$$\frac{29}{3} \quad 7\frac{1}{3}$$

First, convert the improper fraction to a mixed number.

$$\frac{29}{3} = 9\frac{2}{3}$$

Compare the numbers.

$$9\frac{2}{3} > 7\frac{1}{3}$$

$\frac{29}{3}$ is greater than $7\frac{1}{3}$.

**Example 3**

$$4\frac{1}{2} \quad \frac{12}{5}$$

First, change the improper fraction to a mixed number.

$$\frac{12}{5} = 2\frac{2}{5}$$
Then, compare the numbers.

\[
\frac{41}{2} > \frac{22}{5}
\]

\(4\frac{1}{2}\) is greater than \(\frac{22}{5}\).

**Example 4**

\[
\frac{16}{3} \quad \frac{22}{5}
\]

Both fractions are improper. Let’s try comparing the fractions using the lowest common denominator of 3 and 5. The LCD is 15.

First, find the equivalent fraction for each with the denominator of 15.

\[
\frac{16}{3} = \frac{80}{15}
\]
\[
\frac{22}{5} = \frac{66}{15}
\]

Then, compare the fractions.

\[
\frac{80}{15} > \frac{66}{15}
\]

\(\frac{16}{3}\) is greater than \(\frac{22}{5}\).

**Example 5**

\[
\frac{17}{4} \quad 4 \frac{1}{4}
\]

First, convert the mixed fraction to an improper fraction. Multiply the whole number by the denominator and add the numerator. Write it as a fraction over 4.

\(4 \times 4 + 1 = 17\)

\(4\frac{1}{4} = \frac{17}{4}\)

Then, compare the fractions.

\[
\frac{17}{4} = \frac{17}{4}
\]

\(\frac{17}{4}\) is equal to \(4\frac{1}{4}\).

**Review**

Compare each set of values using <, > or =.
4.9. Fraction and Mixed Number Comparison

1. \( \frac{12}{5} \) ___ \( 2 \frac{1}{2} \)
2. \( \frac{16}{4} \) ___ \( 3 \frac{1}{2} \)
3. \( \frac{44}{14} \) ___ \( 6 \frac{1}{2} \)
4. \( \frac{45}{18} \) ___ \( 6 \frac{1}{4} \)
5. \( \frac{18}{12} \) ___ \( 4 \frac{1}{2} \)
6. \( \frac{16}{8} \) ___ \( 2 \)
7. \( \frac{36}{12} \) ___ \( 6 \frac{1}{2} \)
8. \( \frac{99}{33} \) ___ \( 10 \)
9. \( \frac{72}{18} \) ___ \( 8 \frac{2}{3} \)
10. \( \frac{72}{12} \) ___ \( 10 \frac{4}{7} \)
11. \( \frac{80}{20} \) ___ \( 40 \frac{4}{7} \)
12. \( \frac{18}{6} \) ___ \( 25 \)
13. \( \frac{18}{6} \) ___ \( 24 \frac{1}{3} \)
14. \( \frac{78}{4} \) ___ \( 19 \frac{3}{5} \)
15. \( \frac{78}{4} \) ___ \( 10 \frac{8}{9} \)

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.16.

Resources

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/167001
Here you’ll learn when to simplify an expression in the form $M(N + K)$ or $M(N − K)$ by using the Distributive Property.

David and Denise are having an argument. David says that you can’t use the Distributive Property to simplify the expression $\frac{4x + 5}{8}$, while Denise says that you can. Who do you think is right?

**Identifying Expressions Involving the Distributive Property**

The Distributive Property often appears in expressions, and many times it does not involve parentheses as grouping symbols. Recall that the fraction bar acts as a grouping symbol.

**Let’s simplify the following expressions with fractions:**

1. $\frac{9-6y}{3}$
   
   The denominator needs to be distributed to each part of the expression in the numerator.
   
   We can rewrite the expression so that we can see how the Distributive Property should be used:
   
   $$\frac{9-6y}{3} = \frac{1}{3}(9-6y) = \frac{1}{3}(9) - \frac{1}{3}(6y) = 3 - 2y.$$  

2. $\frac{2x+4}{8}$
   
   Think of the denominator as $\frac{1}{8}$: $\frac{2x+4}{8} = \frac{1}{8}(2x + 4)$.
   
   Now apply the Distributive Property: $\frac{1}{8}(2x) + \frac{1}{8}(4) = \frac{2x}{8} + \frac{4}{8}$.
   
   Simplified: $\frac{x}{4} + \frac{1}{2}$.

**Real-World Problems and the Distributive Property**

The Distributive Property is one of the most common mathematical properties seen in everyday life. It crops up in business and in geometry. Anytime we have two or more groups of objects, the Distributive Property can help us solve for an unknown.

**Let’s use the Distributive Property to solve the following problem:**

An octagonal gazebo is to be built as shown below. Building code requires five-foot-long steel supports to be added along the base and four-foot-long steel supports to be added to the roof-line of the gazebo. What length of steel will be required to complete the project?
Each side will require two lengths, one of five and one of four feet respectively. There are eight sides, so here is our equation.

Steel required = 8(4 + 5) feet.

We can use the Distributive Property to find the total amount of steel.

Steel required = 8 \times 4 + 8 \times 5 = 32 + 40 feet.

A total of 72 feet of steel is needed for this project.

Examples

Example 1

Earlier, you were told that David says that you can’t use the Distributive Property to simplify the expression \(\frac{4x + 5}{8}\), while Denise says that you can. Who is correct?
Denise is correct, you can use the Distributive Property to simplify the expression. You can distribute the denominator.

\[
\frac{4x + 5}{8} = \frac{1}{8}(4x + 5) = \frac{1}{8}(4x) + \frac{1}{8}(5) = \frac{x}{2} + \frac{5}{8}
\]

The simplified form of the expression is \(\frac{x}{2} + \frac{5}{8}\).

**Example 2**

Simplify \(\frac{10x+8y-1}{2}\).

First we rewrite the expression so we can see how to distribute the denominator:

\[
\frac{10x + 8y - 1}{2} = \frac{1}{2}(10x + 8y - 1) = \frac{1}{2}(10x) + \frac{1}{2}(8y) - \frac{1}{2}(1) = 5x + 4y - \frac{1}{2}
\]

**Review**

Use the Distributive Property to simplify the following expressions.

1. \((2 - j)(-6)\)
2. \((r + 3)(-5)\)
3. \(6 + (x - 5) + 7\)

Use the Distributive Property to simplify the following fractions.

4. \(\frac{8x+12}{4}\)
5. \(\frac{9x+12}{3}\)
6. \(\frac{11x+12}{2}\)
7. \(\frac{3y+2}{6}\)
8. \(\frac{6-2}{3}\)
9. \(\frac{7-6p}{3}\)

In 10 - 17, write an expression for each phrase.

10. \(\frac{2}{3}\) times the quantity of \(n\) plus 16
11. Twice the quantity of \(m\) minus 3
12. \(-4x\) times the quantity of \(x\) plus 2
13. A bookshelf has five shelves, and each shelf contains seven poetry books and eleven novels. How many of each type of book does the bookcase contain?
14. Use the Distributive Property to show how to simplify \(6(19.99)\) in your head.
15. A student rewrote \(4(9x + 10)\) as \(36x + 10\). Explain the student’s error.
16. Use the Distributive Property to simplify \(9(5998)\) in your head.
17. Amar is making giant holiday cookies for his friends at school. He makes each cookie with 6 oz of cookie dough and decorates each one with macadamia nuts. If Amar has 5 lbs of cookie dough \((1 \text{ lb} = 16 \text{ oz})\) and 60 macadamia nuts, calculate the following.
   a. How many (full) cookies can he make?
   b. How many macadamia nuts can he put on each cookie if each is supposed to be identical?

**Review (Answers)**

To see the Review answers, open this [PDF file](#) and look for section 2.9.
4.11 Coordinate Locations on a Map

In this concept, you will learn how to use coordinates to locate a place on a map.

Ian finds an old map when he’s going through his grandfather’s attic. It seems to be a pirate map of the Spanish Main. On the back of it, he sees scrawled a hasty X and the numbers: 26.5 and 81. He suspects that this map leads to treasure. But where is it?

In this concept, you will learn how to use coordinates to locate a place on a map.

Using Coordinates to Locate Places on Maps

A **coordinate grid** is a grid in which points are graphed. It usually has two or more intersecting lines which divide a plane into quadrants, and in which ordered pairs, or coordinates, are defined.

Maps also use **coordinates**. Around the edges of maps are numbers and sometimes letters. These define the location of cities, states, and other physical entities and landmarks. Most maps use degrees. These define the latitude and longitude.

**Longitude** is the measure, in degrees, of lines vertically on a map. Depending on how the map is made, these lines are sometimes curved.

**Latitude** is the measure, in degrees, of lines horizontally on a map.

These degrees are written as ordered pairs with the latitude listed first and the longitude listed second. Here is an example of a world map with the latitude and longitude lines overlaid on it. For lines to the left of 0° or below the equator, there is in implied negative sign. Sometimes coordinates are also written with directions. 30°S is the latitude line beneath the Tropic of Cancer on the map below.

You can identify different locations on a map if you have the coordinates of the location. For example, which country is located at 30°, 140°?

First, identify the latitude line.
4.11. Coordinate Locations on a Map

In this case, it is below the Tropic of Capricorn line and it goes through Australia and South America. Next, identify the longitude line. In this case, it is off to the right and it seems to go mainly through eastern Europe and Australia. Then, decide where they converge. In this example, they both pass through Australia, so that is the country that exists at those coordinates.

Examples

Example 1

Earlier, you were given a problem about Ian and his pirate treasure. He has found a map with what he suspects are coordinates pointing to pirate treasure, but he doesn’t know where to dig. The coordinates say 26.5 and 81.

First, he decides which coordinate is which. 26.5 must be the latitude because the numbers don’t go that low on the bottom. That means that 81 is the longitude. Next, he locates the latitude and then the longitude of the point. That latitude seems to go through the Atlantic Ocean and some islands. That longitude seems to go up through Cuba and then Florida. Then, he finds where they converge. They seem to meet up in the Florida Keys. He concludes that the booty is in the Keys. He starts dreaming up get-rich-quick schemes that will get him to Florida so he can start digging.
Example 2

In the map above, identify the state found at (40°N, 80°W).

First, find the latitude. This is the horizontal line.

In this case, it is 40°N. This line goes through the middle of the map. It crosses Montana, Illinois, Indiana, Ohio, and Pennsylvania.

Next, find the longitude.

In this case it is 80°W. Starting in South Carolina, it goes up through North Carolina, Virginia, West Virginia, and Pennsylvania.

Then, find the state where they converge.

Pennsylvania is the only state that both lines go through, so it is the answer.

In the following examples, use the map of the world below to identify the countries given by the corresponding latitude and longitudes.
Example 3

\((0^\circ, 60^\circ W)\)

First, find the latitude. This is the horizontal line.
In this case, it is \(0^\circ\), which is also known as the equator.
Next, find the longitude.
In this case it is \(60^\circ W\). It is in what we call the “Western Hemisphere”, and it mostly passes through South America.
Then, find where they converge.
In this case, they converge in Brazil.

Example 4

\((35^\circ N, 90^\circ W)\)

First, find the latitude. This is the horizontal line.
In this case, it is \(35^\circ N\). It is above \(30^\circ N\) in the Northern Hemisphere. It goes through North America, Africa, and Asia.
Next, find the longitude.
In this case it is \(90^\circ W\). It goes mostly through Central and North America.
Then, find where they converge.
In this case, they converge in the United States.

Example 5

\((35^\circ N, 90^\circ E)\)

First, find the latitude. This is the horizontal line.
In this case, it is \(35^\circ N\). It is above \(30^\circ N\) in the Northern Hemisphere. It goes through North America, Africa, and Asia.
Next, find the longitude.
In this case it is \(90^\circ E\). It goes up through Asia.
Then, find where they converge.
In this case, they converge in China.

**Review**

Use a map of the United States to identify each city on the map according to latitude and longitude.

1. What is at 61°, 149°?
2. What is at 30°, 97°?
3. What is at 39°, 71°?
4. What is at 41°, 87°?
5. What is at 41°, 81°?
6. What is at 21°, 157°?
7. What is at 44°, 123°?
8. What is at 30°, 81°?
9. What is at 36°, 115°?
10. What is at 34°, 118°?
11. What is at 35°, 78°?
12. What is at 37°, 77°?
13. What is at 38°, 90°?
14. What is at 27°, 82°?
15. What is at 38°, 77°?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 11.16.

Apply and extend previous understandings of numbers to the system of rational numbers.

6.NS.5 Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

   a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., -(–3) = 3, and that 0 is its own opposite.

   b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

   c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

6.NS.7 Understand ordering and absolute value of rational numbers.

   a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. For example, interpret -3 > -7 as a statement that -3 is located to the right of -7 on a number line oriented from left to right.

   b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. For example, write
-3°C > -7°C to express the fact that -3°C is warmer than -7°C.

c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. For example, for an account balance of -30 dollars, write |−30| = 30 to describe the size of the debt in dollars.

d. Distinguish comparisons of absolute value from statements about order. For example, recognize that an account balance less than -30 dollars represents a debt greater than 30 dollars.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.
5.1 Whole Number Exponents
5.2 Values Written as Powers
5.3 Evaluation and Comparison of Powers
5.4 Expressions with One or More Variables
5.5 Patterns and Expressions
5.6 Words that Describe Patterns
5.7 Single Variable Expressions
5.8 Connect Variable Expressions and the Order of Operations with Real-World Problems
5.9 Numerical Expression Evaluation with Basic Operations
5.10 Calculator Use with Algebraic Expressions
5.11 Numerical Expression Evaluation with Grouping Symbols
5.12 Expression Evaluation with Powers and Grouping Symbols

Apply and extend previous understandings of arithmetic to algebraic expressions.

6.EE.1 Write and evaluate numerical expressions involving whole-number exponents.

6.EE.2 Write, read, and evaluate expressions in which letters stand for numbers.

a. Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation “Subtract y from 5” as 5 - y.

b. Identify parts of an expression using mathematical terms (sum, term, product, factor quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.

c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas V = s^3 and A = 6 s^2 to find the volume and surface area of a cube with sides of length s = 1/2.

6.EE.3 Apply the properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.

6.EE.4 Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.
Sheila’s parents are encouraging her to save her money. Sheila currently has $3. Sheila’s parents tell her that for every month she saves her money instead of spending it, they will double the amount of money she has! Sheila decides she will save her money and not spend it for 6 months. How could Sheila write an expression in exponential form and evaluate to determine how much money she will have in 6 months?

In this concept, you will learn how to write and evaluate expressions in exponential form.

**Exponents**

Sometimes you need to multiply a number or a variable by itself many times.

Here is an example.

\[4 \times 4 \times 4 \times 4 \times 4 \times 4\]

is 4 multiplied by itself 7 times.

To avoid having to write out the 4 again and again, you can use an **exponent**. Whole number **exponents** are shorthand for repeated multiplication of a number by itself.

\[4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^7\]

In this example, 7 is the **exponent** and 4 is the **base**. The **exponent** indicates how many times the **base** is being multiplied by itself.

Using an exponent can also be called “raising to a power”. The exponent represents the power.

For example, \[4^7\] could be read “4 to the seventh power”.

There are two exponents that have special names. A base raised to the power of 2 is said to be **squared**. A base raised to the power of 3 is said to be **cubed**.

Here is an example.
• $4^2$ could be read “4 to the second power” or “4 squared”.
• $4^3$ could be read “4 to the third power” or “4 cubed”.

When you use an exponent to write an expression you are using **exponential form**. $4^7$ is exponential form. When you write out the expression using multiplication without an exponent you are using expanded form. $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$ is expanded form.

Here is an example.

Write the following in exponential form: $6 \times 6 \times 6 \times 6$

First, notice that 6 is being multiplied by itself 4 times. 6 will be your base and 4 will be your exponent.
The answer is $6^4 = 6 \times 6 \times 6 \times 6$.

Here is another example.

Write the following in expanded form and evaluate the expression: $5^3$

First, read the expression. This expression could be read as “5 to the third power” or “5 cubed”.
Next, write the expression in expanded form without an exponent.

$5^3 = 5 \times 5 \times 5$

Then, multiply.

$5 \times 5 \times 5 = 125$

The answer is $5^3 = 5 \times 5 \times 5 = 125$.

**Examples**

**Example 1**

Earlier, you were given a problem about Sheila and her $3. Her parents told her they would double the amount of money she has for every month she saves her money instead of spending it. Sheila decides she will save her money for 6 months and wonders how much money she will have at this point.

First, write an expression to represent how much money Sheila will have after 6 months. Start with how much money she will have after one month and work your way up to 6 months. Remember that to double means to multiply by 2.

• Sheila starts with $3.
  • After 1 month she will have $3 \times 2$.
  • After 2 months she will have $3 \times 2 \times 2$ or $3 \times 2^2$.
  • After 3 months she will have $3 \times 2 \times 2 \times 2$ or $3 \times 2^3$.

Continuing in this pattern you can see that

• After 6 months she will have $3 \times 2^6$.

Next, evaluate the expression in order to figure out how much money she will have in 6 months. First, write the expression in expanded form.

$3 \times 2^6 = 3 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

Now, multiply.

$3 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 192$

The answer is that after 6 months of saving Sheila will have $192. Not bad!
Example 2

Evaluate: $2^3 + 4^2$

First, write the expression in expanded form.

$2^3 + 4^2 = 2 \times 2 \times 2 + 4 \times 4$

Next, multiply each part of the expression.

$2 \times 2 \times 2 + 4 \times 4 = 8 + 16$

Then, add.

$8 + 16 = 24$

The answer is $2^3 + 4^2 = 24$.

Example 3

Write the following in exponential form: $3 \times 3 \times 3 \times 3 \times 3$

First, remember that to write an expression in exponential form you need a base and an exponent. The base is the number that is being multiplied by itself. The exponent is the number of times the base is being multiplied by itself.

The base is 3.
The exponent is 5.

Now, write in exponential form.

$3^5$

The answer is $3 \times 3 \times 3 \times 3 \times 3 = 3^5$.

Example 4

Write the following in expanded form and evaluate the expression: $6^3$

First, write the expression in expanded form without an exponent.

$6^3 = 6 \times 6 \times 6$

Next, multiply.

$6 \times 6 \times 6 = 216$

The answer is $6^3 = 6 \times 6 \times 6 = 216$.

Example 5

Evaluate: $4^3 - 5^2$

First, write the expression in expanded form.

$4^3 - 5^2 = 4 \times 4 \times 4 - 5 \times 5$

Next, multiply each part of the expression.

$4 \times 4 \times 4 - 5 \times 5 = 64 - 25$

Then, subtract.

$64 - 25 = 39$
The answer is $4^3 - 5^2 = 39$.

**Review**

Name the base and exponent in the following expressions. Then, write each in expanded form.

1. $4^5$
2. $3^2$
3. $5^8$
4. $4^3$
5. $6^3$
6. $2^5$
7. $1^{10}$
8. $2^4$
9. $3^4$
10. $5^2$
11. $4^4$
12. $8^{10}$
13. $9^3$
14. $12^2$
15. $13^3$

**Answers for Review Problems**

To see the Review answers, open this [PDF file](#) and look for section 1.7.

**Resources**

Here you’ll learn how to write the product of repeated values using powers.

Cherry is training to be an interior designer. One of her projects is to make notes on every room in which she spends time. Cherry soon gets sick of writing down all the dimensions: 18 feet wide x 18 feet long x 18 feet high. Is there an easier way for Cherry to note down all the same information?

In this concept, you will learn how to write the product of repeated values using powers.

**Exponents**

Factors are a sequence of numbers that are multiplied by each other, like $2 \times 3 \times 4$, where 2, 3, and 4 are factors, or $5 \times 5 \times 5 \times 5$, where 5, 5, 5, and 5 are factors.

Repeated factors can be rewritten as a power using an exponent. Remember, the exponent is the little number that tells you how many times to multiply the base by itself.

Consider this expression:

$$7 \times 7 \times 7 = ____$$

There are three 7’s being multiplied. Since 7 is the number being multiplied, the base is 7. Since the 7 is multiplied by itself 3 times, the exponent (also called the power) is 3.

$$7 \times 7 \times 7 = 7^3$$
Examples

Example 1

Earlier, you were given a problem about Cherry and her lengthy dimensions.
Cherry wants to know an easier alternative to writing 18 feet wide x 18 feet long x 18 feet high.
You now know that when the same number, or factor, is being multiplied repeatedly, you can represent that with an exponent.
In Cherry’s case, the base number is 18, and she is multiplying it 3 times, so 3 is the exponent. There is also a particular way to express the exponent 3: you can say the number is "cubed". In this case, since the units, feet, are multiplied by themselves three times also, the resulting units are "feet cubed" or "cubic feet".
The answer is \((18 \text{ feet})^3\).
Cherry can now make her notations much faster by writing 18\(^3\) ft\(^3\), or 18 cubed, cubic feet.

Example 2

Write the following as a base with an exponent.

\[ 4 \times 4 \times 4 \times 4 \]

First, look at what number is being multiplied by itself, in this case, 4. That number is the base.
Next, write the base, 4, as a full-size number.
Then, write the power (also 4 in this case, since the base is repeated 4 times), as a small number above and to the right of the base.
The answer is \(4^4\).

Example 3

Write the following as a base with an exponent.

\[ 6 \times 6 \times 6 \times 6 \]

First, look at what number is being multiplied by itself. In this case, 6. This is the base.
Next, write the base, 6, as a full-size number.
Then, write the power, 4 in this case, as a small number above and to the right of the base.
The answer is \(6^4\).

Example 4

Write the following as a base with an exponent.

\[ 2 \times 2 \times 2 \times 2 \times 2 \times 2 \]
First, look at what number is being multiplied by itself. In this case, 2. That means 2 is the base. 

Next, write the base, 2, as a full-size number. 

Then, write the power, 7 in this case, as a small number above and to the right of the base. 

The solution is $2^7$.

**Example 5**

Write the following as a base with an exponent.

$$3 \times 3$$

First, look at what number is being multiplied by itself. In this case, 3. This is the base. 

Next, write the base, 3, as a full-size number. 

Then, write the power, 2 in this case, as a small number above and to the right of the base. 

The answer is $3^2$.

**Review**

Write each repeated factor using an exponent.

1. $4 \times 4 \times 4$
2. $3 \times 3 \times 3 \times 3$
3. $2 \times 2$
4. $9 \times 9 \times 9 \times 9$
5. $10 \times 10 \times 10 \times 10 \times 10 \times 10$
6. $1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$
7. $3 \times 3 \times 3 \times 3 \times 3$
8. $4 \times 4$
9. $7 \times 7 \times 7$
10. $6 \times 6 \times 6 \times 6$
11. $11 \times 11 \times 11$
12. $12 \times 12$
13. $18 \times 18 \times 18$
14. $21 \times 21 \times 21 \times 21$
15. $17 \times 17$

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 1.8.

**Resources**
Example Problem
Write each in exponential form.
\[ \frac{2\cdot2\cdot3\cdot3\cdot3}{7\cdot7} = \frac{3^3}{7^2} \]
\[ 12\cdot12\cdot12 \]
Su Chin keeps her turtle, Larry, in a glass tank that is $14^3$ cubic inches. Larry loves to eat and is growing at an alarming rate, and Su Chin will soon need a larger tank for him. She looks at an online pet store and sees that they have two tanks for sale: one is 2,744 cubic inches, and the other is 4,096 cubic inches. How can Su Chin compare the size of her tank to the new tanks to ensure she buys a larger one?

In this concept, you will learn how to evaluate and compare powers.

Powers

Let’s consider this expression.

$4^2$

The large number, 4, is the base. The base is the number to be multiplied by itself.

The small number, 2, is the exponent. The exponent tells you how many times to multiply the base by itself. The exponent is also known as a power.

Once you know the base and exponent, you can evaluate the expression. Evaluating an expression with an exponent means to complete the indicated multiplication and write the result as a product.

Consider the expression above, $4^2$.

First, identify the base, in this case 4.

Then, identify the exponent, in this case 2.

Next, write the expression as the base multiplied by itself the number of times indicated by the exponent:

$4^2 = 4 \times 4$
Finally, perform the multiplication.

\[4^2 = 4 \times 4 = 16\]

**NOTE**: A common error related to exponents is to multiply the base by the exponent. Don’t do that! Clearly, since \(4^2 = 4 \times 4 = 16\) and \(4 \times 2 = 8\), they aren’t the same thing.

You can also compare the values of powers using symbols: > (greater than), < (less than) and = (equal to).

To compare the values of different powers, you will need to evaluate each power and then compare them.

Let’s look at an example.

Evaluate \(1^{100}\)

This looks a bit intimidating, since the power is so large. However, since the base is 1, and regardless how many times you multiply 1 by itself, it is always 1, this is actually not difficult at all.

The answer is: \(1^{100} = 1 \times 1 \times 1 \ldots \times 1 = 1\)

In fact, 1 to any power is equal to 1.

**Examples**

**Example 1**

Earlier, you were given a problem about Su Chin and her rapidly growing reptile.

Su Chin needs to compare her current tank, which is \(14^3\) cubic inches, with two possible replacements that measure 2,744 cubic inches, and 4,096 cubic inches, in order to choose the largest tank.

First, evaluate the power of Su Chin’s current tank.

\[14^3 = 14 \times 14 \times 14 = 2,744\]

The answer is 2,744 cubic inches.

Then, compare those dimensions to the other tanks, using <, >, or =.

\[2,744 = 2,744\]
\[2,744 < 4,096\]

The answer is the second tank is largest.

The first tank is the same size as Su Chin’s current tank, at 2,744 cubic inches. The second tank is greater in size, at 4,096 cubic inches.

Su Chin can now buy the second tank in order to upgrade Larry’s home to a spacious 4,096 cubic inches.

**Example 2**

Insert >, <or = in the blank.

\[4^5 \quad \square \quad 5^4\]

First, evaluate each number.
Example 3
Evaluate the expression $2^6$.
First, identify the base, in this case 2.
Then, identify the exponent, in this case 6.
Next, consider the expression as the base multiplied by itself the number of times indicated by the exponent:

\[2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2\]

Finally, perform the multiplication.

\[2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64\]

The answer is 64.

Example 4
Evaluate the expression $6^3$.
First, identify the base, in this case 6.
Then, identify the exponent, in this case 3.
Next, consider the expression as the base multiplied by itself the number of times indicated by the exponent:
Finally, perform the multiplication.

\[6^3 = 6 \times 6 \times 6\]

The answer is 216.

**Example 5**

Which is greater? \(2^7 \quad 5^3\)

First, evaluate each number.

\[2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2\]
\[= 4 \times 2 \times 2 \times 2 \times 2 \times 2\]
\[= 8 \times 2 \times 2 \times 2 \times 2\]
\[= 16 \times 2 \times 2 \times 2\]
\[= 32 \times 2 \times 2\]
\[= 64 \times 2\]
\[= 128\]

\[5^3 = 5 \times 5 \times 5\]
\[= 25 \times 5\]
\[= 125\]

Finally, compare the values. 128 > 125.

The answer is \(2^7 > 5^3\).

**Review**

Evaluate each expression.

1. \(2^2\)
2. \(3^2\)
3. \(6^2\)
4. \(7^3\)
5. \(8^4\)
6. \(2^6\)
7. \(3^5\)
5.3. Evaluation and Comparison of Powers

8. \(6^4\)
9. \(5^3\)
10. \(1^{100}\)

Compare each power using \(<\), \(>\), or \(=\):

11. \(4^2 \quad 2^4\)
12. \(3^2 \quad 1^5\)
13. \(6^3 \quad 3^6\)
14. \(7^2 \quad 5^2\)
15. \(8^3 \quad 9^2\)

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 1.9.

**Resources**

![Image](https://www.ck12.org/flix/render/embeddedobject/176113)
Here you’ll be given an expression containing one or more variables and values for the variables. You’ll learn how to plug the values into the expression and simplify the expression.

Suppose you know the area of a circle is approximately $3.14r^2$, where $r$ is the radius of the circle. What if a circle has a radius of 25 inches? How would you find its area?

### Expressions with One or More Variables

Just like in the English language, mathematics uses several words to describe one thing. For example, sum, addition, more than, and plus all mean to add numbers together. To **evaluate** means to follow the verbs in the math sentence. **Evaluate** can also be called simplify or answer. To begin to evaluate a mathematical **expression**, you must first **substitute** a number for the variable. To **substitute** means to replace the variable in the sentence with a value.

Now try out your new vocabulary.

#### Let’s evaluate

Evaluate means to follow the directions, which is to take 7 times $y$ and subtract 11. Because $y$ is the number 4, we can evaluate our expression as follows:

\[
7 \times 4 - 11 \\
28 - 11 \\
17
\]

The solution is 17.

Because algebra uses variables to represent the unknown quantities, the multiplication symbol $\times$ is often confused with the variable $x$. To help avoid confusion, mathematicians replace the multiplication symbol with parentheses ( ) or the multiplication dot $\cdot$, or by writing the expressions side by side.

#### Rewrite

\[P = 2 \times l + 2 \times w\]

can be written as \[P = 2 \cdot l + 2 \cdot w.\]

It can also be written as \[P = 2l + 2w.\]

#### Apply expressions to a real life situation:

To prevent major accidents or injuries, these horses must be fenced in a rectangular pasture. If the dimensions of the pasture are 300 feet by 225 feet, how much fencing should the ranch hand purchase to enclose the pasture?
Begin by drawing a diagram of the pasture and labeling what you know.

![Diagram of the pasture](https://www.ck12.org/flx/render/embeddedobject/179475)

To find the amount of fencing needed, you must add all the sides together:

\[ L + L + W + W \]

By substituting the dimensions of the pasture for the variables \( L \) and \( W \), the expression becomes:

\[ 300 + 300 + 225 + 225 \]

Now we must evaluate by adding the values together. The ranch hand must purchase \( 300 + 300 + 225 + 225 = 1,050 \) feet of fencing.
### Examples

#### Example 1

Earlier, you were asked how to find the area of a circle with a radius of 25 inches if you know the area of a general circle is approximately $3.14r^2$ with $r$ being the radius of the circle.


The area of the circle is approximately 1962.5 in$^2$.

#### Example 2

Write the expression $2 \times a$ in a more condensed form and then evaluate it for $3 = a$.

$2 \times a$ can be written as $2a$. We can substitute 3 for $a$:

$$2(3) = 6$$

#### Example 3

If it costs $9.25 for a movie ticket, how much does it cost for 4 people to see a movie?

Since each movie ticket is $9.25, we multiply this price by the 4 people buying tickets to get the total cost:

$$9.25 \times 4 = 37.00$$

It costs $37 for 4 people to see a movie.

### Review

In 1-4, write the expression in a more condensed form by leaving out the multiplication symbol.

1. $2 \times 11x$
2. $1.35 \cdot y$
3. $3 \times \frac{1}{4}$
4. $\frac{1}{4} \cdot z$

In 5-9, evaluate the expression.

5. $5m + 7$ when $m = 3$
6. $\frac{1}{3}(c)$ when $c = 63$
7. $8.15(h)$ when $h = 40$
8. $(k - 11) \div 8$ when $k = 43$
9. $(−2)^2 + 3(j)$ when $j = −3$

In 10-17, evaluate the expression. Let $a = −3$, $b = 2$, $c = 5$, and $d = −4$.

10. $2a + 3b$
11. $4c + d$
12. $5ac - 2b$
13. $\frac{2a}{c+d}$
14. $\frac{3b}{d}$
5.4. Expressions with One or More Variables

15. \[
\frac{a\cdot4b}{3c+2d}
\]
16. \[
\frac{1}{a+b}
\]
17. \[
\frac{ab}{cd}
\]

In 18-25, evaluate the expression. Let \(x = -1\), \(y = 2\), \(z = -3\), and \(w = 4\).

18. \(8x^3\)
19. \(\frac{5x^2}{6w^3}\)
20. \(3z^2 - 5w^2\)
21. \(x^2 - y^2\)
22. \(\frac{x^2 + w^3}{y^3}\)
23. \(2x^2 - 3x^2 + 5x - 4\)
24. \(4w^3 + 3w^2 - w + 2\)
25. \(3 + \frac{1}{z}\)

In 26-30, evaluate the expression in each real-life problem.

26. The measurement around the widest part of these holiday bulbs is called their circumference. The formula for circumference is \(2(r)\pi\), where \(\pi \approx 3.14\) and \(r\) is the radius of the circle. Suppose the radius is 1.25 inches. Find the circumference.

27. The dimensions of a piece of notebook paper are 8.5 inches by 11 inches. Evaluate the writing area of the paper. The formula for area is length \(\times\) width.
28. Sonya purchased 16 cans of soda at $0.99 each. What is the amount Sonya spent on soda?
29. Mia works at a job earning $4.75 per hour. How many hours should she work to earn $124.00?
30. The area of a square is the side length squared. Evaluate the area of a square with a side length of 10.5 miles.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.2.
5.5 Patterns and Expressions

Here you’ll learn how to take an English phrase and produce an equivalent algebraic expression. You’ll also practice taking an algebraic expression and producing an equivalent English phrase.

Jeremy read that degrees Celsius converted to degrees Fahrenheit is "the sum of 32 and \( \frac{9}{5} \) times the temperature in degrees Celsius." However, he’s not sure how to convert this into an algebraic expression. What do you think an equivalent algebraic expression would be?

Writing Algebraic Expressions

In mathematics, especially in algebra, we look for patterns in the numbers that we see. Using mathematical verbs and variables, expressions can be written to describe a pattern. Recall that an algebraic expression is a mathematical phrase combining numbers and/or variables using mathematical operations. We can describe patterns using phrases as well, and we want to be able to translate these phrases into algebraic expressions.

Consider a theme park charging an admission of $28 per person. A rule can be written to describe the relationship between the amount of money taken at the ticket booth and the number of people entering the park. In words, the relationship can be stated as “The money taken in dollars is (equals) twenty-eight times the number of people who enter the park.”

The English phrase above can be translated (written in another language) into an algebraic expression. Using mathematical verbs and nouns, any phrase can be written as an algebraic expression.

Let’s write an algebraic expression for the each of the following phrases:

1. The product of \( c \) and 4.

   The verb is product, meaning “to multiply.” Therefore, the phrase is asking for the answer found by multiplying \( c \) and 4. The nouns are the number 4 and the variable \( c \). The expression becomes \( 4 \times c \), which may also be written as \( 4c \), or \( 4c \).

2. The phrase about the theme park from above: The money taken in dollars is (equals) twenty-eight times the number of people who enter the park.
An appropriate variable to describe the number of people could be \( p \). Rewriting the English phrase into a mathematical phrase, it becomes \( 28 \times p \).

3. 5 less than 2 times a number.

Some phrases are harder to translate than others. The word "less" lets you know that you are going to take away, or subtract, a number. Many students will want to turn this expression into \( 5 - 2n \), but this is not what the phrase is telling us. Whatever the value of "2 times a number "or \( 2n \), we want to write an expression that shows we have 5 less than that. That means that we need to subtract 5 from \( 2n \). The correct answer is \( 2n - 5 \).

**Examples**

**Example 1**

Earlier, you were asked told that the degrees Fahrenheit temperature is "the sum of 32 and \( \frac{9}{5} \) times the temperature in degrees Celsius". What is an equivalent algebraic expression?

We can use \( c \) to represent the degrees Celsius. The word "sum" indicates addition and the word "times" indicates multiplication. The multiplication between \( c \) and \( \frac{9}{5} \) occurs first and then 32 is added to the result. Therefore the expression to represent the phrase is:

\[
32 + \frac{9}{5}c
\]

**Example 2**

A student organization sells shirts to raise money for events and activities. The shirts are printed with the organization’s logo and the total costs are $100 plus $7 for each shirt. The students sell the shirts for $15 each. Write an expression for the cost and an expression for the revenue (money earned).
We can use \( x \) to represent the number of shirts. For the cost, we have a fixed $100 charge and then $7 times the number of shirts printed. This can be expressed as \( 100 + 7x \). For the revenue, we have $15 times the number of shirts sold, or \( 15x \).

**Review**

For exercises 1 - 15, translate the English phrase into an algebraic expression. For the exercises without a stated variable, choose a letter to represent the unknown quantity.

1. Sixteen more than a number
2. The quotient of \( h \) and 8
3. Forty-two less than \( y \)
4. The product of \( k \) and three
5. The sum of \( g \) and \( -7 \)
6. \( r \) minus 5.8
7. 6 more than 5 times a number
8. 6 divided by a number minus 12
9. A number divided by \( -11 \)
10. 27 less than a number times four
11. The quotient of 9.6 and \( m \)
12. 2 less than 10 times a number
13. The quotient of \( d \) and five times \( s \)
14. 35 less than \( x \)
15. The product of 6, \( -9 \), and \( u \)

In exercises 16 - 24, write an English phrase for each algebraic expression

16. \( J - 9 \)
17. \( \frac{n}{14} \)
18. 17 - \( a \)
19. 31 - 16
20. \( \frac{1}{7} (h)(b) \)
21. \( \frac{b}{3} + \frac{2}{7} \)
22. 4.7 - 2\( f \)
23. 5.8 + \( k \)
24. \( 2l + 2w \)

In exercises 25 - 28, define a variable to represent the unknown quantity and write an expression to describe the situation.

25. The unit cost represents the quotient of the total cost and number of items purchased. Write an expression to represent the unit cost of the following: The total cost is $14.50 for \( n \) objects.
26. The area of a square is the side length squared.
27. The total length of ribbon needed to make dance outfits is 15 times the number of outfits.
28. What is the remaining amount of chocolate squares if you started with 16 and have eaten some?

Use your sense of variables and operations to answer the following questions.

29. Describe a real-world situation that can be represented by \( h + 9 \).
30. What is the difference between \( \frac{7}{m} \) and \( \frac{m}{7} \)?
Review (Answers)

To see the Review answers, open this PDF file and look for section 1.6.
Here you’ll learn how to interpret the figures in a table by writing an English sentence or an algebraic expression. You’ll also use the algebraic expressions you find to make predictions about the future.

Many bicyclists have biking watches that are able to record the time spent biking and the distance traveled. They are able to download the data recorded by their watches to a computer and view a table with times in minutes in one column and distances in miles in another column. How could they use this data to write an English sentence or an algebraic expression?

Using Words to Describe Patterns

Sometimes patterns are given in tabular format (meaning presented in a table). An important job of data analysts is to describe a pattern so others can understand it.

Let’s describe the following patterns in words:

1. We can see from the table that \( y \) is five times bigger than \( x \). Therefore, the pattern is that the “\( y \) value is five times larger than the \( x \) value.”

2. Zarina has a \$100 gift card and has been spending money in small regular amounts. She checks the balance on the card at the end of every week and records the balance in the following table. Using the table, describe the pattern in words and in an expression.

<table>
<thead>
<tr>
<th>Week #</th>
<th>Balance ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>78</td>
</tr>
<tr>
<td>2</td>
<td>56</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
</tr>
</tbody>
</table>

Each week the amount of her gift card is \$22 less than the week before. The pattern in words is: “The gift card started at \$100 and is decreasing by \$22 each week.” As we saw in the last lesson, this sentence can be translated into the algebraic expression

\[ 100 - 22w \]

Using Expressions

Note that the expression found in the second problem can be used to answer questions and predict the future. Suppose, for instance, that Zarina wanted to know how much she would have on her gift card after 4 weeks if she
used it at the same rate. By substituting the number 4 for the variable \( w \), it can be determined that Zarina would have $12 left on her gift card.

\[
100 - 22w
\]

When \( w = 4 \), the expression becomes:

\[
100 - 22(4) \\
100 - 88 \\
12
\]

After 4 weeks, Zarina would have $12 left on her gift card.

Examples

Example 1

Earlier, you were told that bicyclists are able to download the their data on the time they spent biking and the distance they traveled. This data can be represented as a table with the times in minutes in one column and the distances in miles in another column. How can the bicyclists use this data to write an English sentence or an algebraic expression?

As shown in this section, tables are useful for finding patterns. To write a sentence or expression about the data, the bicyclists must look at the data and find a pattern for how the distance changes as time increases.

Example 2

Jose starts training to be a runner. When he starts, he can run 3 miles per hour. After 5 weeks of training, Jose can run faster. After each week, he records his average speed while running. He summarizes this information in the following table:


**Write an expression for Jose’s increased speed and predict how fast he will be able to run after 6 weeks.**

We will use \( w \) to represent the number of weeks. Jose’s speed starts at 3 mph, and from the table we can see that it increases by 0.25 miles per hour every week. This gives us the expression \( 3 + 0.25w \). Now we substitute in \( w = 6 \) and get the following:

\[
3 + 0.25(6) = 3 + 1.5 = 4.5
\]

If Jose keeps up his training, by the end of the 6th week, he should be able to run 4.5 miles per hour.

**Review**

In questions 1-3, write the pattern of the table: a) in words and b) with an algebraic expression.

1. **Number of workers and number of video games packaged**

<table>
<thead>
<tr>
<th>People</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>0</td>
<td>65</td>
<td>87</td>
<td>109</td>
<td>131</td>
<td>153</td>
<td>175</td>
</tr>
</tbody>
</table>

2. **The number of hours worked and the total pay**

<table>
<thead>
<tr>
<th>Hours</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Pay</td>
<td>15</td>
<td>22</td>
<td>29</td>
<td>36</td>
<td>43</td>
<td>50</td>
</tr>
</tbody>
</table>

3. **The number of hours of an experiment and the total number of bacteria**

<table>
<thead>
<tr>
<th>Hours</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bacteria</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>32</td>
<td>1024</td>
</tr>
</tbody>
</table>

4. **With each filled seat, the number of people on a Ferris wheel doubles.**

   1. Write an expression to describe this situation.
   2. How many people are on a Ferris wheel with 17 seats filled?
   3. Using the theme park situation from the lesson, how much revenue would be generated by 2,518 people?
Mixed Review

6. Use parentheses to make the equation true: \(10 + 6 \div 2 - 3 = 5\).
7. Find the value of \(5x^2 - 4y\) for \(x = -4\) and \(y = 5\).
8. Find the value of \(\frac{x^2y}{x+y}\) for \(x = 2\) and \(y = -4\).
9. Simplify: \(2 - (t - 7)^2 \times (u^2 - v)\) when \(t = 19, u = 4,\) and \(v = 2\).

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.7.
<Expression>

An expression is a mathematical phrase that contains numbers and operations.

Here are some examples of expressions:

- $3x + 10$
- $-15 + 7 - 1$
- $5^2 - 1$
- $15r + 2$
A variable is a symbol or letter (such as $x, m, R, y, P,$ or $a$) that is used to represent a quantity that might change in value. A variable expression is an expression that includes variables. Another name for a variable expression is an algebraic expression.

Here are some examples of variable expressions:

- $3x + 10$
- $10r$
- $b^3 + 2$
- $mx - 3$

A single variable expression is a variable expression with just one variable in it.

You can use a variable expression to describe a real world situation where one or more quantities has an unknown value or can change in value.

To evaluate a variable expression means to find the value of the expression for a given value of the variable. To evaluate, substitute the given value for the variable in the expression and simplify using the order of operations. To follow the order of operations, you always need to do any multiplication/division first before any addition/subtraction.

Here is an example.

Evaluate the expression $10k - 44$ for $k = 12$.

First, remember that when you see a number next to a letter, like “$10k$”, it means to multiply.

Next, substitute 12 in for the letter $k$ in the expression.

$10(12) - 44$

Notice that you can put parentheses around the 12 to keep it separate from the number 10.

Now, simplify the expression using the order of operations. You will need to multiply first and then subtract.

$10(12) - 44 = 120 - 44 = 76$

The answer is 76.

Here is another example that involves division.

Evaluate the expression $\frac{x}{3} + 2$ for $x = 24$.

First, remember that a fraction bar is like a division sign. $\frac{x}{3}$ is the same as $x \div 3$.

Next, substitute 24 in for the letter $x$ in the expression.

$\frac{24}{3} + 2$

Now, simplify the expression using the order of operations. You will need to divide first and then add.

$\frac{24}{3} + 2 = 8 + 2 = 10$

The answer is 10.
Examples

Example 1

Earlier, you were given a problem about Shelly and her bracelet business. Shelly is selling bracelets this summer and her profit for selling $b$ bracelets is given by the expression $4b - 150$. Shelly wants to calculate what her profit will be if she sells 50 bracelets.

To calculate her profit from selling 50 bracelets, Shelly needs to evaluate the expression $4b - 150$ for $b = 50$.

First, substitute 50 in for the letter $b$ in the expression.

$4(50) - 150$

Now, simplify the expression using the order of operations. You will need to multiply first and then subtract.

$4(50) - 150 = 200 - 150$
$= 50$

Shelly’s profit from selling 50 bracelets would be $50.

Example 2

Evaluate $\frac{x}{7} - 5$ if $x$ is 49.

First, substitute 49 in for the letter $x$ in the expression.

$\frac{49}{7} - 5$

Now, simplify the expression using the order of operations. You will need to divide first and then subtract.

$\frac{49}{7} - 5 = 7 - 5$
$= 2$

The answer is 2.

Example 3

Evaluate $4x - 9$ if $x$ is 20.

First, substitute 20 in for the letter $x$ in the expression.

$4(20) - 9$

Now, simplify the expression using the order of operations. You will need to multiply first and then subtract.

$4(20) - 9 = 80 - 9$
$= 71$

The answer is 71.
**Example 4**

Evaluate $5y + 6$ if $y$ is 9.

First, substitute 9 in for the letter $y$ in the expression.

$5(9) + 6$

Now, simplify the expression using the order of operations. You will need to multiply first and then add.

\[
5(9) + 6 = 45 + 6 = 51
\]

The answer is 51.

**Example 5**

Evaluate $\frac{a}{4} - 8$ if $a$ is 36.

First, substitute 36 in for the letter $a$ in the expression.

$\frac{36}{4} - 8$

Now, simplify the expression using the order of operations. You will need to divide first and then subtract.

\[
\frac{36}{4} - 8 = 9 - 8 = 1
\]

The answer is 1.

**Review**

Evaluate each expression if the given value of $r$ is 9.

1. $\frac{r}{3}$
2. $63 - r$
3. $11r$
4. $2r + 7$
5. $3r + r$
6. $4r - 2r$
7. $r + 5r$
8. $12r - 1$

Evaluate each expression for $h = 12$.

9. $70 - 3h$
10. $6h + 6$
11. $4h - 9$
12. $11 + \frac{h}{4}$
13. \(3h + h\)
14. \(2h + 5h\)
15. \(6h - 2h\)

**Answers for Review Problems**

To see the Review answers, open this [PDF file](https://www.ck12.org/flx/render/embeddedobject/165855) and look for section 1.4.

**Resources**
Connect Variable Expressions and the Order of Operations with Real-World Problems

In this concept, you will learn how to write a real world problem as a variable expression and how to evaluate the expression you have written using the order of operations.

For her birthday, Sara received a ticket to go to the ballet to see The Nutcracker. Sara needs to figure out the cost for the tickets for her friends, but she isn’t sure exactly how many of her friends will be going, it could be three or four. The cost for one ticket is $35.00, and there is a $2.00 one-time fee for the ticket purchase. Sara needs to know the price if she buys three tickets and also if she buys four tickets. If Sara could write a variable expression to model her problem, how could she use it to figure out the cost of the tickets?

In this concept, you will learn how to write a real world problem as a variable expression and how to evaluate the expression you have written using the order of operations.

Variable Expressions and Order of Operations

When you have an unknown quantity in a problem, you can use a variable to represent the unknown quantity and then use it to evaluate the problem. In the real world there are many scenarios to which you can apply what you know about variable expressions and evaluating them.

Let’s look at an example.

Many events require that you purchase a ticket for admission. Tickets can have various prices if tickets are sold for adults, seniors or children. If you know the number and type of tickets being purchased and the price for each ticket, then you can figure out the total cost of the tickets.

An amusement park charges each person eight dollars admission and one dollar and fifty-cents per ride. Write a variable expression to model this situation and use it to find the cost of admission and five ride tickets.

First, notice there is a cost of eight dollars for admission. You can start with the number $8.00

Next, notice there is a charge of $1.50 per ride. Therefore you would have to add 1.50 x, where $x$ is the number of rides, to the admission cost.

Then, write a variable expression you can use to determine the cost of going to the amusement park and enjoying the rides.

\[8.00 + 1.50x\]

Remember the variable $x$ represents the number of rides. You can now evaluate the variable expression to find the cost of going to the amusement park and going on five rides.

First substitute $x = 5$ into $8.00 + 1.50x$ and write the new expression.

\[8.00 + 1.50(5)\]

Use the order of operations to evaluate the expression.
Multiply: $1.50 \times 5 = 7.50$ to clear parenthesis and write the new expression.

$$8.00 + 7.50$$

Next, add: $8.00 + 7.50 = $15.50

The answer is $15.50.

Let’s look at one more problem.

An ice cream cone costs $3.50 plus an additional $1.25 for chocolate dip. Write an expression to model the cost of two chocolate dip ice cream cones.

You must write a numerical expression to model the real-world problem. Since there are two cones and two dips, use $2(3.50 + 1.25)$.

Now use the order of operations to evaluate the expression.

First, perform the indicated operation inside the parenthesis.

Next, add $3.50 + 1.25 = 4.75$, and write the new expression.

$$2(4.75)$$

Then, multiply: $2 \times 4.75 = 9.50$

The answer is $9.50.

**Examples**

**Example 1**

Earlier, you were given a problem about Sara and her ballet tickets. She needs to know the prices of a three-ticket purchase and a four-ticket purchase. The price of one ticket is $35.00 and there is a $2.00 processing fee.

Write a variable expression to determine the cost of the tickets.

First, determine the variable. The variable will be the number of tickets purchased since that is the changeable amount. Let $x$ be the variable.

Remember the price of $35.00 of must be included with the variable.

Next, write this part of the expression $35x$.

Then, include the onetime fee of $2.00 and write the variable expression.

The answer is $35x + 2$.

Sara needs to use the variable expression and the order of operations to calculate the cost of 3 tickets and of 4 tickets.

First, substitute $x = 3$ into $35x + 2$ and write the new expression.

$$35(3) + 2$$

Next, multiply: $35 \times 3 = 105$ to clear parenthesis and write the new expression.
105 + 2

Then, add: 105 + 2 = $107.00
The answer is $107.00.

First, substitute $x = 4$ into $35x + 2$ and write the new expression.

$35(4) + 2$

Next, multiply: $35 \times 3 = 140$ to clear parentheses and write the new expression.

$140 + 2$

Then, add: $140 + 2 = $142.00
The answer is $142.00.

Sara will have to pay $107.00 for three tickets or $142.00 for four tickets.

**Example 2**

Harriet is making an almond cake for her sister and needs some ingredients. When she arrives at the store, Harriet sees that a can of evaporated milk costs $1.99 and almonds are $3.99 per pound.

Write a variable expression to model this problem so that Harriet can calculate her total cost based on how many pounds of almonds she purchases. Use your variable expression to calculate the cost of her purchase if Harriet buys 3 pounds of almonds.

First, determine the variable. The variable is going to be the number of pounds of almonds purchased since that is the changeable amount. Let $x$ be the variable.

Remember the price of $3.99 must be included with the variable, write this part of the expression: $3.99x$.

Harriet isn’t only purchasing almonds; she is also buying a can of evaporated milk. You must include this cost in your variable expression.

Add the cost of the can of milk to the first part of your expression:

$3.99x + 1.99$

The answer is $3.99x + 1.99$.

First, substitute $x = 3$ into $3.99x + 1.99$ and write the new expression.

$3.99(3) + 1.99$

Use the order of operations to evaluate the expression.

Next, multiply: $3.99 \times 3 = 11.97$ to clear parenthesis and write the new expression.

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\[ 11.97 + 1.99 \]

Then, add: \(11.97 + 1.99 = 13.96\)
The answer is $13.96.

**Example 3**

The cost for three people to fly from Sydney, Nova Scotia to Calgary, Alberta is $1535.75. The airline also applies an additional charge of $25.00 for each piece of checked luggage. Write an expression to model the problem.

You must write a variable expression to model the problem.

\[ 1535.75 + 25x, \text{ where } x \text{ represents the number of pieces of checked luggage.} \]

Use your variable expression to figure out the total cost of the trip if 5 pieces of luggage are checked.

First, substitute \(x = 5\) into \(1535.75 + 25x\) and write the new expression.

\[ 1535.75 + 25(5) \]

Use the order of operations to evaluate the expression.

Next, multiply: \(25 \times 5 = 125\) to clear parenthesis and write the new expression.

\[ 1535.75 + 125 \]

Then, add: \(1535.75 + 125 = 1660.75\)
The answer is $1660.75.

**Example 4**

Maria wants to sew a new valence for her living room window. The window used to be one pane of glass 5 feet wide but the new window has a center section that is 5 feet wide and two smaller sections on either side. The new sections are each 3 feet long. The length of the valence must be double the width of the window. Write a variable expression to model the problem.

\[ 3(5 + 2x), \text{ where } x \text{ represents the width of two smaller sections of the window.} \]

Since the width of each of the smaller window sections is 3 three feet, evaluate the variable expression when \(x = 3\).

First, substitute the value \(x = 3\) into the expression.

\[ 3(5 + 2(3)) \]

Use the order of operations to evaluate the variable expression.

Next, perform the operations inside the parenthesis.

Multiply \(2(3) = 6\), and write the new expression.
5.8. Connect Variable Expressions and the Order of Operations with Real-World Problems

\[ 3(5 + 6) \]

Next, add: \( 5 + 6 = 11 \) and write the new expression.

\[ 3(11) \]

Then, multiply \( 3(11) = 33 \) to clear the parenthesis.
The answer is 33.

**Review**

Write a variable expression for each situation described below.

1. A pound of apples costs $4.50. Kelly isn’t sure how many pounds she is going to purchase.
2. Each touch down at a football game is worth 7 points. Write a variable expression where the number of touchdowns scored can change.
3. Alex bought several bunches of bananas. Each bunch costs $.89. Write a variable expression to describe this situation.
4. If an ice cream cone costs $3.20 and hot fudge is an additional $.45, write a variable expression where the number of ice cream cones can change.
5. There are six children in the Smith family. If each child gets a haircut, write a variable expression to show the cost of the haircut so that the total cost can be calculated.
6. A turkey costs $6.75 per pound. Write a variable expression where the weight of the turkey is the changeable amount.
7. The car wash charges $15.00 per car. Write a variable expression that can be used to calculate the number of cars washed in one hour.
8. Kelly bought a pair of sneakers for $35.00. She also bought a pile of different laces. Each set of laces costs $3.00. Write a variable expression to show how Kelly could calculate her total cost.

Now go back to each of the variable expressions that you have written as the answers for the above questions and evaluate each expression using 4 as the given value for the variable. These are your answers for numbers 9-16.

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 1.5.
5.9 Numerical Expression Evaluation with Basic Operations

In this concept, you will learn how to evaluate numerical expressions using the four operations.

Margot volunteers at a local bird sanctuary. She just spent two weeks on vacation, so on her first day back she is talking to her supervisor about what she has missed. When Margot left, there were 256 birds. In her absence, 3 birds have each given birth to 5 baby birds, 2 birds were released into the wild and 3 new injured birds were brought in. How can Margot write an expression to figure out the new sanctuary population?

In this concept, you will learn how to evaluate numerical expressions using the four operations.

**Order of Operations**

An equation is a number sentence that describes two values that are equal to each other. The values are separated by the "equals" sign, e.g. $3 + 4 = 7$. In this case, 3 + 4 and 7 are equal entities. Sometimes you will be given an incomplete equation, which you have to "solve" in order to make both sides equal.

An expression is a number sentence without an equals sign. It can be simplified and/or evaluated, but it cannot be "solved".

Expressions can involve more than one operation, so it is important to know which operation to do first. The order of operations tells you the correct order for completing operations within an expression.

<table>
<thead>
<tr>
<th>Order of Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>P - parentheses</td>
</tr>
<tr>
<td>E - exponents</td>
</tr>
<tr>
<td>MD - multiplication or division, in order from left to right</td>
</tr>
<tr>
<td>AS - addition or subtraction, in order from left to right</td>
</tr>
</tbody>
</table>
Let’s look at an example.

\[ 4 + 3 \times 5 \]

This is an expression with both addition and multiplication. The order of operations tells you that multiplication comes before addition. So, first multiply and then add.

\[
\begin{align*}
4 + 3 \times 5 &= 4 + 15 \\
&= 19
\end{align*}
\]

When you evaluate this expression correctly, using order of operations, the answer is 19.

If you had NOT followed the order of operations, and instead evaluated from left to right, it would look like this.

\[
\begin{align*}
4 + 3 \times 5 &= 7 \times 5 \\
&= 35
\end{align*}
\]

You would have gotten an incorrect answer. That is why it is important to always follow the order of operations.

Examples

Example 1

Earlier, you were given a problem about Margot and her bountiful birds.

Margot needs to write and evaluate an expression to see how the original population of 256 has changed since 3 birds each gave birth to 5 baby birds, 2 birds were released, and 3 injured birds were brought in.

First, identify the important information in the problem.

256 birds to start
3 birds have 5 babies each
2 birds released
3 birds added

Next, write this as an expression.

\[ 256 + 3 \times 5 - 2 + 3 \]

Then, use order of operations to evaluate the expression.
256 + 3 \times 5 - 2 + 3 \quad \text{Multiply } 3 \times 5
\hphantom{256 + 3 \times 5 - 2 + 3} \quad 256 + 15 - 2 + 3 \quad \text{Add } 256 + 15
\hphantom{256 + 3 \times 5 - 2 + 3} \quad 271 - 2 + 3 \quad \text{Subtract } 271 - 2
\hphantom{256 + 3 \times 5 - 2 + 3} \quad 269 + 3 \quad \text{Add } 269 + 3
\hphantom{256 + 3 \times 5 - 2 + 3} \quad 272

The answer is 272.
Margot's sanctuary now has 272 birds.

**Example 2**

Evaluate the expression using order of operations.

\[
6 + 8 \times 4 - 11 + 6 = \text{___}
\]

First, complete the multiplication from left to right.

\[
6 + 32 - 11 + 6
\]

Then, complete the addition and subtraction from left to right.

\[
6 + 32 - 11 + 6
\quad 38 - 11 + 6
\quad 27 + 6
\quad 33
\]

The answer is 33.

**Example 3**

Evaluate the following expression.

\[
8 - 1 \times 4 + 3 = \text{___}
\]

First, perform the multiplication:
5.9. Numerical Expression Evaluation with Basic Operations

\[ 8 - 1 \times 4 + 3 \quad \text{Multiply } 1 \times 4 = 4 \]
\[ 8 - 4 + 3 \]

Next, add and subtract from left to right:

\[ 8 - 4 + 3 \quad \text{Subtract } 8 - 4 = 4 \]
\[ 4 + 3 \quad \text{Add } 4 + 3 = 7 \]
\[ 7 \]

The solution is 7.

**Example 4**

Evaluate the following expression.

\[ 5 + 9 \times 3 - 6 + 2 = \_\_\_ \]

First, perform multiplication and division from left to right:

\[ 2 \times 6 + 8 \div 2 \quad \text{Multiply } 2 \times 6 \]
\[ 12 + 8 \div 2 \quad \text{Divide } 8 \div 2 \]
\[ 12 + 4 \]

Next, add:

\[ 12 + 4 \quad \text{Add } 12 + 4 \]
\[ 16 \]

The answer is 16.

**Example 5**

Evaluate the following expression.

\[ 5 + 9 \times 3 - 6 + 2 = \_\_\_ \]
First, perform multiplication and division from left to right:

\[
5 + 9 \times 3 - 6 + 2 \quad \text{Multiply } 9 \times 3 \\
5 + 27 - 6 + 2
\]

Next, add and subtract from left to right:

\[
5 + 27 - 6 + 2 \quad \text{Add } 5 + 27 \\
32 - 6 + 2 \quad \text{Subtract } 32 - 6 \\
26 + 2 \quad \text{Add } 26 + 2 \\
28
\]

The answer is 28.

**Review**

Evaluate each expression according to the correct order of operations.

1. \(2 + 3 \times 4 + 7 = \) 
2. \(4 + 5 \times 2 + 9 - 1 = \) 
3. \(6 \times 7 + 2 \times 3 = \) 
4. \(4 \times 5 + 3 \times 1 - 9 = \) 
5. \(5 \times 3 \times 2 + 5 - 1 = \) 
6. \(4 + 7 \times 3 + 8 \times 2 = \) 
7. \(9 - 3 \times 1 + 4 - 7 = \) 
8. \(10 + 3 \times 4 + 2 - 8 = \) 
9. \(11 \times 3 + 2 \times 4 - 3 = \) 
10. \(6 + 7 \times 8 - 9 \times 2 = \) 
11. \(3 + 4^2 - 5 \times 2 + 9 = \) 
12. \(2^2 + 5 \times 2 + 6^2 - 11 = \) 
13. \(3^2 \times 2 + 4 - 9 = \) 
14. \(6 + 3 \times 2^2 + 7 - 1 = \) 
15. \(7 + 2 \times 4 + 3^2 - 5 = \) 

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 1.10.

**Resources**
5.10 Calculator Use with Algebraic Expressions

Here you’ll learn how to type an expression into your calculator and evaluate it, either by entering the value(s) of the variable(s) directly into the expression, or by storing the value(s) of the variable(s) in your calculator’s memory and then entering the expression.

What if you wanted to evaluate the expression \( \frac{x^2 + 4x + 3}{2x - 9x - 5} \) when \( x = 7 \)? If you had your calculator handy, it could make things easier, but how would you enter an expression like this into your calculator? Also, how would you tell your calculator that \( x = 7 \)?

Using a Calculator to Evaluate Expressions

A calculator, especially a graphing calculator, is a very useful tool in evaluating algebraic expressions. A graphing calculator follows the order of operations, PEMDAS. In this section we will explain two ways of evaluating expressions with a graphing calculator.

Method 1 This method is the direct input method. After substituting all values for the variables, you type in the expression, symbol for symbol, into your calculator.

Method 2 This method uses the STORE function of the Texas Instrument graphing calculators, such as the TI-83, TI-84, or TI-84 Plus.

Let’s use a graphing calculator to solve the following problems:

1. Evaluate \( [3(x^2 - 1)^2 - x^4 + 12] + 5x^3 - 1 \) when \( x = -3 \).

First, try Method 1:

Substitute the value \( x = -3 \) into the expression.

\[ [3((-3)^2 - 1)^2 - (-3)^4 + 12] + 5(-3)^3 - 1 \]

Now, type it directly into the calculator as shown below:

The potential error here is that you may forget a sign or a set of parentheses, especially if the expression is long or complicated. Make sure you check your input before writing your answer. An alternative is to type in the expression in appropriate chunks - do one set of parentheses, then another, and so on.

To compare, let’s also try Method 2:
First, store the value \( x = -3 \) in the calculator. Type \(-3 \text{[STO]} x\). (The letter \( x \) can be entered using the \( x\)-[VAR] button or \([\text{ALPHA]} + \text{[STO]}\)). Then type in the expression in the calculator and press \([\text{ENTER}]\).

![Calculator Image](URL: https://www.ck12.org/flx/render/embeddedobject/182916)

The answer is \(-13\).

Note: On graphing calculators there is a difference between the minus sign and the negative sign. When we stored the value negative three, we needed to use the negative sign, which is to the left of the \([\text{ENTER}]\) button on the calculator. On the other hand, to perform the subtraction operation in the expression we used the minus sign. The minus sign is right above the plus sign on the right.

2. Evaluate the expression: \(\frac{3x^2-4y^2+x^4}{(x+y)^2}\) for \( x = -2, y = 1 \).

Store the values of \( x \) and \( y \): \(-2\text{[STO]} x\), \( 1 \text{[STO]} y\). The letters \( x \) and \( y \) can be entered using \([\text{ALPHA]} + \text{[KEY]}\). Input the expression in the calculator. When an expression shows the division of two expressions, be sure to use parentheses: (numerator) \( ÷ \) (denominator). Press \([\text{ENTER}]\) to obtain the answer \(-.88\) or \(-\frac{8}{9}\).
Examples

Example 1

Earlier, you were asked to evaluate the expression $\frac{x^2 + 4x + 3}{2x^2 - 9x - 5}$ when $x = 7$. To tell the calculator that $x = 7$, store the value of $x$.

Type in 7 [STO] $x$ to store the value of $x$. The letter $x$ can be entered using [ALPHA]+[KEY]. Then input the expression in the calculator. Make sure to use parentheses to group the expressions in the numerator and denominator together. Press [ENTER] to obtain the answer 2.6 or $\frac{8}{3}$.

Example 2

Evaluate the expression $\frac{x - y}{x^2 + y^2}$ for $x = 3, y = -1$.

Store the values of $x$ and $y$: 3[STO] $x$, -1 [STO] $y$. The letters $x$ and $y$ can be entered using [ALPHA] + [KEY]. Input the expression in the calculator. When an expression shows the division of two expressions, be sure to use parentheses: (numerator) ÷ (denominator). Press [ENTER] to obtain the answer 0.4.

Review

In 1-5, evaluate each expression using a graphing calculator.

1. $x^2 + 2x - xy$ when $x = 250$ and $y = -120$
2. $(xy - y^4)^2$ when $x = 0.02$ and $y = -0.025$
3. $\frac{x+y-2}{xy+3x+4z}$ when $x = \frac{1}{2}, y = \frac{3}{2},$ and $z = -1$
4. \(\frac{(x+y)^2}{4x^2-y^2}\) when \(x = 3\) and \(y = -5\)

5. The formula to find the volume of a spherical object (like a ball) is \(V = \frac{4}{3}(\pi)r^3\), where \(r\) = the radius of the sphere. Determine the volume for a grapefruit with a radius of 9 cm.

In 6-9, insert parentheses in each expression to make a true equation.

6. \(5 - 2 \cdot 6 - 4 + 2 = 5\)
7. \(12 \div 4 + 10 - 3 \cdot 3 + 7 = 11\)
8. \(12 - 8 - 4 \cdot 5 = -8\)
9. Let \(x = -1\). Find the value of \(-9x + 2\).

**Mixed Review**

10. The area of a trapezoid is given by the equation \(A = \frac{h}{2}(a+b)\). Find the area of a trapezoid with bases \(a = 10\) cm, \(b = 15\) cm, and height \(h = 8\) cm.

![Trapezoid](image)

11. The area of a circle is given by the formula \(A = \pi r^2\). Find the area of a circle with radius \(r = 17\) inches.

![Circle](image)

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 1.5.
Doug is collecting all the ribbon he can find in his house. He receives three 6 ft blue ribbon rolls from his mom, finds 4 ft of orange ribbon in a drawer, and grabs 7 ft of gold ribbon from his little sister, who then comes over with scissors and takes back 2 ft for herself. Doug sits at his desk and starts to measure all his collected ribbon with a measuring tape. Is there an easier way for Doug to figure out how many feet of ribbon he has?

In this concept, you will learn how to evaluate numerical expressions using powers and grouping symbols.

Order of Operations

Parentheses are symbols that group things together. This becomes very important in numerical expressions, because operations inside parentheses are always completed first when evaluating the expression. Let’s review the order of operations.

Order of Operations

- parenthens
- exponents
- multiplication or division, in order from left to right
- addition or subtraction, in order from left to right

You can see that, according to the order of operations, parentheses come first.

Let’s see how this works.

\[ 2 + (3 - 1) \times 2 \]
In this problem, there are four elements to consider. You have one set of parentheses, addition, subtraction and multiplication. You can evaluate this expression using the order of operations. Here is what the process looks like:

\[ 2 + (3 - 1) \times 2 \]
\[ 2 + 2 \times 2 \]
\[ 2 + 4 \]
\[ = 6 \]

The answer is 6.

Let’s consider a different problem, and take it step by step:

\[ 35 + 3^2 - (3 \times 2) \times 7 \]

First, evaluate parentheses.

\[ 35 + 3^2 - 6 \times 7 \]

Next, evaluate exponents.

\[ 35 + 9 - 6 \times 7 \]

Then, complete multiplication in order from left to right.

\[ 35 + 9 - 42 \]

Finally, complete addition and/or subtraction in order from left to right.

\[ 44 - 42 = 2 \]

The answer is 2.

**Examples**

**Example 1**

Earlier, you were given a problem about Doug and his ribbon pile.

Doug needs to figure out the total length of ribbon he has collected.

He receives 3 of the 6 ft blue ribbon rolls from his mom, finds 4 ft of orange ribbon in a drawer, and grabs 7 ft of gold ribbon from his little sister, who then comes over with scissors and takes back 2 ft for herself.

First, identify the important information.
receives 3 6 ft rolls
finds 4 ft
takes 7 ft
gives back 2 ft
Next, write this as an expression.

\[ 3 \times 6 + 4 + 7 - 2 \]

Then, use order of operations to evaluate the expression.

\[
\begin{align*}
3 \times 6 + 4 + 7 - 2 & \quad \text{Multiply } 3 \times 6 = 18 \\
18 + 4 + 7 - 2 & \quad \text{Add } 18 + 4 = 22 \\
22 + 7 - 2 & \quad \text{Add } 22 + 7 = 29 \\
29 - 2 & \quad \text{Subtract } 29 - 2 = 27 \\
27 & \quad \text{Final Answer}
\end{align*}
\]

The answer is 27.
Doug has collected 27 feet of ribbon with which to make mischief.

**Example 2**

Evaluate the following expression.

\[ 7^3 - 3^2 + 15 \times 2 + (2 + 3) \]

First, follow the order of operations and evaluate the parentheses and exponents.

\[
\begin{align*}
7^3 & = 7 \times 7 \times 7 = 343 \\
3^2 & = 3 \times 3 = 9 \\
(2 + 3) & = 5
\end{align*}
\]

Next, substitute these values back into the original number sentence.

\[ 343 - 9 + 15 \times 2 + 5 \]

Then, complete the multiplication.
$15 \times 2 = 30$

Finally, complete the addition and subtraction in order from left to right.

$343 - 9 + 30 + 5$

$369$

The answer is 369.

**Example 3**

Evaluate the following expression.

$16 + 2^3 - 5 + (3 \times 4)$

First, evaluate the operations inside of parenthesis.

$16 + 2^3 - 5 + (3 \times 4) \quad \text{Evaluate } 3 \times 4 = 12$

Next, evaluate the exponents.

$16 + 2^3 - 5 + 12 \quad \text{Evaluate } 2^3 = 8$

$16 + 8 - 5 + 12$

Then, complete the addition and subtraction from left to right.

$16 + 8 - 5 + 12 \quad \text{Add } 16 + 8 = 24$

$24 - 5 + 12 \quad \text{Subtract } 24 - 5 = 19$

$19 + 12 \quad \text{Add } 19 + 12 = 31$

$31$

The answer is 31.
**Example 4**

Evaluate the following expression.

\[ 9^2 + 2^2 - 5 \times (2 + 3) \]

First, evaluate the parenthesis.

\[ 9^2 + 2^2 - 5 \times (2 + 3) \quad \text{Evaluate} \ 2 + 3 = 5 \]

\[ 9^2 + 2^2 - 5 \times 5 \]

Next, evaluate the exponents.

\[ 9^2 + 2^2 - 5 \times 5 \quad \text{Evaluate} \ 9^2 = 81 \]

\[ 81 + 2^2 - 5 \times 5 \quad \text{Evaluate} \ 2^2 = 4 \]

\[ 81 + 4 - 5 \times 5 \]

Then, multiply.

\[ 81 + 4 - 5 \times 5 \quad \text{Multiply} \ 5 \times 5 = 25 \]

\[ 81 + 4 - 25 \]

Finally, complete the addition and subtraction operations from left to right.

\[ 81 + 4 - 25 \quad \text{Add} \ 81 + 4 = 85 \]

\[ 85 - 25 \quad \text{Subtract} \ 85 - 25 = 60 \]

\[ 60 \]

The solution is 60.

**Example 5**

Evaluate the following expression.

\[ 8^2 \div 2 + 4 - 1 \times 6 \]

First, evaluate the exponent.
8^2 ÷ 2 + 4 − 1 × 6  Evaluate 8^2 = 64
64 ÷ 2 + 4 − 1 × 6

Then, divide and multiply from left to right.

64 ÷ 2 + 4 − 1 × 6  Divide 64 ÷ 2 = 32
32 + 4 − 1 × 6  Multiply 1 × 6 = 6
32 + 4 − 6

Next, add and subtract from left to right.

32 + 4 − 6  Add 32 + 4 = 36
36 − 6  Subtract 36 − 6 = 30
30

The answer is 30.

Review

Evaluate each expression according to the order of operations.

1. 3 + (2 + 7) − 3 + 5 = ___
2. 2 + (5 − 3) + 7^2 − 11 = ___
3. 4 × 2 + (6 − 4) − 9 + 5 = ___
4. 8^2 − 4 + (9 − 3) + 12 = ___
5. 7^3 − 100 + (3 + 4) − 9 = ___
6. 7 + (3^2 + 7) − 11 + 5 = ___
7. 2^4 + (8 + 7) + 13 − 5 = ___
8. 3 × 2 + (2^2 + 7) − 11 + 15 = ___
9. 8 + (6 + 7) − 2 × 3 = ___
10. 22 + (3^4 + 7) − 73 + 15 = ___
11. 3^2 + (4^2 − 7) − 3 + 25 = ___
12. 6^3 + (3^2 + 17) − 73 + 4 = ___
13. 243 − (5^3 + 27) − 83 + 9 = ___
14. 72 + (11^2 + 117) − 193 + 75 = ___
15. 82 + (10^2 + 130) − 303 + 115 = ___

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.11.
Resources

MEDIA
Click image to the left or use the URL below.
URL: https://www.ck12.org/flix/render/embeddedobject/176120
For her best friend’s birthday, Jean bought a huge birthday hat. It’s too big to actually wear, so her plan is to add a base to it, buy fabric, and cover it with fabric. Jean needs to know how much fabric to buy. Jean’s older brother told her that to find the approximate area of the surface of the birthday hat she can use the following formula:

\[ 3.14 \left( \frac{d}{2} \right)^2 + 1.57dl \]

where \( d \) is the diameter of the base of the hat and \( l \) is the slant height of the hat.
Jean measured her birthday hat and found that the diameter is 15 inches and the slant height is 18 inches. How can Jean use the formula that her brother gave her to find the surface area of the birthday hat so that she can buy the right amount of fabric?

In this concept, you will learn how to evaluate variable expressions involving exponents and parentheses.

### Evaluating Expressions with Powers

An expression is a mathematical phrase that contains numbers and operations. A variable expression is an expression that contains variables.

To evaluate a variable expression means to find the value of the expression for given values of the variables. To evaluate, substitute the given values for the variables in the expression and simplify using the order of operations (PEMDAS).

Remember that the order of operations are:

1. Parentheses: Start by simplifying any part of the expression in parentheses using the order of operations.
2. Exponents: Rewrite any terms that contain exponents without exponents.
3. Multiplication/Division: Do any multiplication and/or division in order from left to right.
4. Addition/Subtraction: Do any addition and/or subtraction in order from left to right.

Here is an example.

Evaluate the expression $2x^2 - (x + 7)$ if $x = 4$.

First, substitute 4 in for $x$ in the expression.

$2(4)^2 - (4 + 7)$

Next, notice that this expression has multiplication, exponents, subtraction, parentheses, and addition. According to the order of operations, you must start by simplifying the parts of the expression in parentheses. $(4)$ in the first part of the expression is already simplified. $(4 + 7)$ in the second part of the expression still needs to be simplified.

$2(4)^2 - (4 + 7) = 2(4)^2 - 11$

Next, continue to follow the order of operations and evaluate the exponent. Note that only the 4 is squared. Remember that $4^2$ means $4 \cdot 4$ which is 16.

$2(4)^2 - 11 = 2(16) - 11$

Now, multiply.

$2(16) - 11 = 32 - 11$

Finally, subtract.

$32 - 11 = 21$

The answer is 21.
**Example 1**

Earlier, you were given a problem about Jean and the birthday hat she bought for her friend’s birthday. She wants to find the surface area of the hat so she can buy the correct amount of fabric to cover it. She got the following formula from her brother:

\[ 3.14 \left( \frac{d}{2} \right)^2 + 1.57dl \]

where \( d \) is the diameter of the base of the hat and \( l \) is the slant height of the hat. Jean measured the birthday hat and found that the diameter is 15 inches and the slant height is 18 inches.

To find the surface area, she should first substitute 15 in for \( d \) and 18 in for \( l \) in the formula.

\[ 3.14 \left( \frac{15}{2} \right)^2 + 1.57(15)(18) \]

Next, notice that this expression has multiplication, division, exponents, parentheses, and addition. According to the order of operations, you must start by simplifying the parts of the expression in parentheses that need to be simplified. Start by simplifying the \( \left( \frac{15}{2} \right) \) at the beginning of the expression.

\[ 3.14 \left( \frac{15}{2} \right)^2 + 1.57(15)(18) = 3.14(7.5)^2 + 1.57(15)(18) \]

Next, continue to follow the order of operations and evaluate the exponent. Use your calculator to help you calculate \( 7.5^2 = 7.5 \cdot 7.5 \).

\[ 3.14(7.5)^2 + 1.57(15)(18) = 3.14(56.25) + 1.57(15)(18) \]

Now, multiply from left to right. You can use your calculator to help.

\[ 3.14(56.25) + 1.57(15)(18) = 176.625 + 1.57(15)(18) = 176.625 + 23.55(18) = 176.625 + 423.9 \]

Finally, add.

\[ 176.625 + 423.9 = 600.525 \]

The answer is that the surface area of the hat (including the base that she added on) is about 600.525 square inches. With that knowledge, Jean knows exactly how much fabric she needs to buy to help decorate the birthday hat!
Example 2

Evaluate $6x^2 - 2x + (x + 3)$ if $x$ is 4.

First, substitute 4 in for $x$ in the expression.

$$6(4)^2 - 2(4) + (4 + 3)$$

Next, notice that this expression has multiplication, exponents, subtraction, parentheses, and addition. According to the order of operations, you must start by simplifying the parts of the expression in parentheses that need to be simplified. Start by simplifying the $(4 + 3)$ at the end of the expression.

$$6(4)^2 - 2(4) + (4 + 3) = 6(4)^2 - 2(4) + 7$$

Next, continue to follow the order of operations and evaluate the exponent.

$$6(4)^2 - 2(4) + 7 = 6(16) - 2(4) + 7$$

Now, multiply from left to right.

$$6(16) - 2(4) + 7 = 96 - 2(4) + 7$$

$$= 96 - 8 + 7$$

Finally, add and subtract from left to right. You will subtract first since that comes first in the expression.

$$96 - 8 + 7 = 88 + 7$$

$$= 95$$

The answer is 95.

Example 3

Evaluate $3x^2 - 2 + (x + 3)$ for $x = 2$.

First, substitute 2 in for $x$ in the expression.

$$3(2)^2 - 2 + (2 + 3)$$

Next, notice that this expression has multiplication, exponents, subtraction, parentheses, and addition. According to the order of operations, you must start by simplifying the parts of the expression in parentheses that need to be simplified. Start by simplifying the $(2 + 3)$ at the end of the expression.

$$3(2)^2 - 2 + (2 + 3) = 3(2)^2 - 2 + 5$$
Next, continue to follow the order of operations and evaluate the exponent.

\[ 3(2)^2 - 2 + 5 = 3(4) - 2 + 5 \]

Now, multiply.

\[ 3(4) - 2 + 5 = 12 - 2 + 5 \]

Finally, add and subtract from left to right. You will subtract first since that comes first in the expression.

\[ 12 - 2 + 5 = 10 + 5 = 15 \]

The answer is 15.

**Example 4**

Evaluate \( \frac{24}{x} + (9 - x) + y^2 \) for \( x = 3 \) and \( y = 4 \).

First, substitute 3 in for \( x \) and 4 in for \( y \) in the expression.

\[ \frac{24}{3} + (9 - 3) + 4^2 \]

Next, notice that this expression has division, addition, parentheses, subtraction, and exponents. According to the order of operations, you must start by simplifying the parts of the expression in parentheses that need to be simplified. Start by simplifying the \((9 - 3)\) in the middle of the expression.

\[ \frac{24}{3} + 6 + 4^2 \]

Next, continue to follow the order of operations and evaluate the exponent.

\[ \frac{24}{3} + 6 + 4^2 = \frac{24}{3} + 6 + 16 \]

Now, divide. Remember that a fraction bar means division so \( \frac{24}{3} \) is the same as \( 24 \div 3 \).

\[ \frac{24}{3} + 6 + 16 = 8 + 6 + 16 \]

Finally, add from left to right.

\[ 8 + 6 + 16 = 14 + 16 = 30 \]

The answer is 30.
Example 5

Evaluate $5x^2 - 2 + (3 + 3)$ if $x$ is 5.

First, substitute 5 in for the $x$ in the expression.

$$5(5^2) - 2 + (3 + 3)$$

Next, notice that this expression has multiplication, exponents, subtraction, parentheses, and addition. According to the order of operations, you must start by simplifying the parts of the expression in parentheses that need to be simplified. Start by simplifying the $(3 + 3)$ at the end of the expression.

$$5(5^2) - 2 + 6 = 5(25) - 2 + 6$$

Next, continue to follow the order of operations and evaluate the exponent.

$$5(25) - 2 + 6 = 125 - 2 + 6$$

Now, multiply.

$$125 - 2 + 6 = 123 + 6 = 129$$

The answer is 129.

Review

Evaluate the following variable expressions for $x = 4, y = 2, z = 3$.

1. $x^2 + y$
2. $2y^2 + 5 - 2$
3. $x^2 - y^2 + z$
4. $3x^2 + 2x^2$
5. $8 + x^2 - 4y$
6. $14 ÷ 2 + z^2 - y$
7. $20 + z^2 - y$
8. $5x - 2y + 3z$
9. $5 + (x - z) + 5(6)$

10. $8 + x - y^2 + z$
11. $(x + y) + 5 \cdot 2 - 3$
12. $4x^2 + 3z^3 + 2$

Decide whether each of the following statements are true or false.

13. Parentheses are a grouping symbol.
14. Exponents can’t be evaluated unless the exponent is equal to 3.
15. If there is multiplication and division in a problem you always do the multiplication first.

**Answers for Review Problems**

To see the Review answers, open this PDF file and look for section 1.10.

**Resources**

Apply and extend previous understandings of arithmetic to algebraic expressions.

6.EE.1 Write and evaluate numerical expressions involving whole-number exponents.

6.EE.2 Write, read, and evaluate expressions in which letters stand for numbers.

a. Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation “Subtract y from 5” as $5 - y$.

b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression $2 \cdot (8 + 7)$ as a product of two factors; view $(8 + 7)$ as both a single entity and a sum of two terms.

c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas $V = s^3$ and $A = 6 \cdot s^2$ to find the volume and surface area of a cube with sides of length $s = 1/2$.

6.EE.3 Apply the properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression $3 \cdot (2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6 \cdot (4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.

6.EE.4 Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). For example, the expressions $y + y + y$ and $3y$ are equivalent because they name the same number regardless of which number $y$ stands for.
Chapter Outline

6.1 Single Variable Equations from Verbal Models
6.2 Inequalities that Describe Patterns
6.3 Addition and Subtraction Phrases as Equations
6.4 Multiplication and Division Phrases as Equations
6.5 Inequality Expressions

Reason about and solve one-variable equations and inequalities.

6.EE.5 Understand solving an equation or inequality as a process of answering a question: Which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which $p$, $q$, and $x$ are all nonnegative rational numbers.

6.EE.8 Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.
Kelvin has twice as many chickens in his chicken coop as Murray has in his. If Kelvin has 60 chickens in his coop, write an equation to represent \( c \), the number of chickens in Murray’s chicken coop.

In this concept, you will learn to write single variable equations from verbal models.

**Writing Single Variable Equations from Verbal Models**

Changing a word problem into an equation can often help you in solving the problem. Carefully read the question to determine if the equation has addition, subtraction, multiplication, or division.

Here is an example.

Write “four times a number is twelve” as an equation.

First, let \( x \) be “a number.”

Next, consider that the word “times” means you multiply.

\[
4 \times x
\]

\[
4x
\]
Then, consider that the word “is” means equals.

\[4x \text{ is } 12\]

The equation is \(4x = 12\).

Here is another example.

Write “seven less than a number is fourteen” as an equation.

First, let \(x\) be “a number.”

\[7 \text{ less than } x \text{ is } 14\]

Next, consider that “less than” means subtraction, but be careful about the order.

\[x - 7 \text{ is } 14\]

Then, consider that the word “is” means equals.

\[x - 7 = 14\]

The equation is \(x - 7 = 14\).

**Examples**

**Example 1**

Earlier, you were given a problem about the chicken coops.

Kelvin has 60 chickens, which is twice as many chickens as Murray has. How can Kelvin write a single variable equation to represent his number of chickens?

First, simplify the language.

Kelvin has 60 chickens and this is 2 times \(c\), the number of chickens Murray has.

Next, consider that “is” means equals, and “times” means multiplication.

\[
\begin{align*}
60 &= 2 \times c \\
60 &= 2c
\end{align*}
\]

The equation \(60 = 2c\) represents the number of chickens in Murray’s coop.

**Example 2**

Carrie made 3 liters of lemonade for a party. After the party, she had 0.5 liters of lemonade left. Write an equation to represent \(n\), the number of liters of lemonade that her guests drank.
Use a number, an operation sign, a variable, or an equal sign to represent each part of the problem. Since the question tells you how many liters of lemonade were left after the party, this will be a subtraction equation.

Carrie started with 3 liters, and \( n \) is the number of liters that the guest drank. So, \( 3 - n \) is how much lemonade there was after the party, 0.5 liters.

For this problem, it may help to write an equation in simplified words and then translate those words into an algebraic equation.

\[
\text{(number of liters made)} - \text{(number of liters guests drank)} = \text{(number of liters left)}
\]

\[
3 - n = 0.5
\]

The equation is \( 3 - n = 0.5 \).

**Example 3**

Write an equation for the following phrase: six and a number is twenty.

First, let \( x \) be “a number.”

\[
6 + x \text{ is } 20
\]

Next, consider that “and” means addition.

\[
6 + x = 20
\]

Then, consider that the word “is” means equals.

\[
6 + x = 20
\]

The equation is \( 6 + x = 20 \).

**Example 4**

Write an equation for the following phrase: eighteen divided by a number is three.

First, let \( x \) be “a number.”

\[
18 \div x \text{ is } 3
\]

Next, you know this involves division.

\[
\frac{18}{x} \text{ is } 3
\]

Then, consider that the word “is” means equals.
The equation is \( \frac{18}{x} = 3 \).

**Example 5**

Write an equation for the following phrase: five times a number is twenty-five.

First, let \( y \) be “a number.”

\[ 5 \times y = 25 \]

Next, consider that the word “times” means you multiply.

\[ 5y = 25 \]

Then, consider that the word “is” means equals.

\[ 5y = 25 \]

The equation is \( 5y = 25 \).

**Review**

Write an equation for each verbal model.

1. Ten times a number is thirty.
2. Five times a number is fifteen.
3. A number and seven is eleven.
4. A number divided by three is twelve.
5. A number and eighteen is thirty.
6. A number divided by twelve is fourteen.
7. Seven times a number is forty-nine.
8. A number divided by thirteen is seven.
9. Eight times a number is equal to sixty-four.

Write an algebraic expression for each situation below.

10. Arturo has 8 fewer stickers in his collection than Julissa has in hers. Let \( j \) represent the number of stickers in Julissa’s collection. Write an expression to represent the number of stickers in Arturo’s collection.
11. Let \( c \) represent the number of cookies on a plate. Three friends share all the cookies on the plate equally. Write an expression to represent the number of cookies each friend has after they are shared equally.
12. Carly is twice as old as her sister. Let $s$ represent her sister’s age in years. Write an expression to represent Carly’s age in years.

13. The length of a rectangle is 3 inches longer than its width. Let $w$ stand for the width in inches. Write an expression to represent the length in inches.

Write an algebraic equation for each word problem below.

14. The chorus teacher divides all the students in the chorus into 3 equal groups. Each of the groups has 6 students in it. Write an equation that could be used to represent $n$, the total number of students in the chorus.

15. Matt’s dog weighs 30 pounds. His dog weighs 20 pounds more than his cat. Write an equation that could be used to represent $c$, the weight, in pounds, of Matt’s cat.

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 7.3.
Here you will learn how to read about a real-life situation and write an inequality that represents this situation. You will then solve the inequality and plug the answer back into the inequality to check your work.

Consider that you are driving a car at 45 miles per hour and you know that your destination is less than 150 miles away. What inequality could you set up to solve for the number of hours that you have left to travel? After you’ve solved the inequality, how could you check to make sure that your answer is correct?

**Inequalities**

In some cases there are multiple answers to a problem or the situation requires something that is not exactly equal to another value. When a mathematical sentence involves something other than an equal sign, an *inequality* is formed.

An **algebraic inequality** is a mathematical sentence connecting an expression to a value, a variable, or another expression with an inequality sign.

Listed below are the most common inequality signs:

- > “greater than”
- ≥ “greater than or equal to”
- ≤ “less than or equal to”
- < “less than”
- ≠ “not equal to”

Below are several examples of inequalities:

\[
3x < 5 \quad x^2 + 2x - 1 > 0 \quad \frac{3x}{4} \geq \frac{x}{2} - 3 \quad 4 - x \leq 2x
\]

**Let’s translate the following statement into an inequality:**

Avocados cost $1.59 per pound. How many pounds of avocados can be purchased for less than $7.00?

Choose a variable to represent the number of pounds of avocados purchased, say \(a\).

\[1.59(a) < 7\]
Checking a Solution to an Inequality

Unlike equations, inequalities have more than one solution. However, you can check whether a value, such as $x = 6$, is a solution to an inequality the same way as you would check if it is the solution to an equation—by substituting it in and seeing if you get a true algebraic statement.

Now, let's check the solution for the following inequalities:

1. Is $m = 11$ a solution set to $4m + 30 \leq 70$?

   Plug in $m = 11$, to see if we get a true statement.

   
   \[
   4(11) + 30 \leq 70 \\
   44 + 30 \leq 70 \\
   74 \leq 70
   \]

   Since $m = 11$ gives us a false statement, it is not a solution to the inequality.

2. Is $m = 10$ a solution to $4m + 30 \leq 70$?

   Substitute in $m = 10$:

   
   \[
   4(10) + 30 \leq 70 \\
   40 + 30 \leq 70 \\
   70 \leq 70
   \]

   For $70 \leq 70$ to be a true statement, we need $70 < 70$ or $70 = 70$. Since $70 = 70$, this is a true statement, so $m = 10$ is a solution.
Examples

Example 1

Earlier, you were told that you are driving a car at 45 miles per hour and your destination is less than 150 miles away. What inequality could you set up to solve for the number of hours that you have left to travel? How can you check to make sure that your answer is correct?

Choose a variable to represent the number of hours left to travel, say \( h \).

If you travel 45 miles per hour for \( h \) hours, then the expression \( 45h \) represents the total number of miles you traveled. This value is less than 150 so the inequality that represents the situation is:

\[
45h < 150
\]

To check a solution to this equation, you would substitute in the value and make sure that the statement is valid. For example, let’s check that \( h = 3 \) is a solution. Substituting, you get:

\[
45(3) < 150
\]

This is a true statement since 135 is less than 150. \( h = 3 \) is a solution to this inequality. You could have 3 hours left to travel.

Example 2

Check whether \( x = 3 \) is a solution to \( 2x - 5 < 7 \).

Substitute in \( x = 3 \), to see if it is a solution to \( 2x - 5 < 7 \).
2(3) − 5 < 7
6 − 5 < 7
1 < 7

Since 1 is less than 7, we have a true statement, so \(x = 3\) is a solution to \(2x - 5 < 7\).

**Example 3**

Check whether \(x = 6\) is a solution to \(2x - 5 < 7\).

Check if \(x = 6\) is a solution to \(2x - 5 < 7\).

\[
2(6) - 5 < 7 \\
12 - 5 < 7 \\
7 < 7
\]

Since 7 is not less than 7, this is a false statement. Thus \(x = 6\) is not a solution to \(2x - 5 < 7\).

**Review**

1. Define solution.
2. What is the difference between an algebraic equation and an algebraic inequality? Give an example of each.
3. What are the five most common inequality symbols?

In 4-7, define the variables and translate the following statements into algebraic equations.

4. A bus can seat 65 passengers or fewer.
5. The sum of two consecutive integers is less than 54.
6. An amount of money is invested at 5% annual interest. The interest earned at the end of the year is greater than or equal to $250.
7. You buy hamburgers at a fast food restaurant. A hamburger costs $0.49. You have at most $3 to spend. Write an inequality for the number of hamburgers you can buy.

For exercises 8-11, check whether the given solution set is the solution set to the corresponding inequality.

8. \(x = 12\); \(2(x + 6) \leq 8x\)
9. \(z = -9\); \(1.4z + 5.2 > 0.4z\)
10. \(y = 40\); \(-\frac{3}{5}y + \frac{1}{2} < -18\)
11. \(t = 0.4\); \(80 \geq 10(3t + 2)\)

In 12-14, find the solution set.

12. Using the burger and French fries situation from the previous Concept, give three combinations of burgers and fries your family can buy without spending more than $25.00.
13. Solve the avocado inequality from Example A and check your solution.
14. On your new job you can be paid in one of two ways. You can either be paid $1000 per month plus 6% commission on total sales or be paid $1200 per month plus 5% commission on sales over $2000. For what amount of sales is the first option better than the second option? Assume there are always sales over $2000.

Mixed Review

15. Translate into an algebraic equation: 17 less than a number is 65.
16. Simplify the expression: $3^4 \div (9 \times 3) + 6 - 2$.
17. Rewrite the following without the multiplication sign: $A = \frac{1}{2} \cdot b \cdot h$.
18. The volume of a box without a lid is given by the formula $V = 4x(10 - x)^2$, where $x$ is a length in inches and $V$ is the volume in cubic inches. What is the volume of the box when $x = 2$?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.9.
In this concept, you will learn how to write addition and subtraction phrases as single variable equations.

Sal leads an informal cycling team of six people. He wants to register them for a race, but doesn’t know if there is enough space. The maximum number of allowed racers on the course is 138 cyclists. What is the maximum number of cyclists that can already be registered if the whole team can join the race? How can Sal write a single variable equation to represent this problem?

In this concept, you will learn how to write addition and subtraction phrases as single variable equations.

**Writing Addition and Subtraction Phrases as Equations**

An expression connects numbers and/or variables with operations, such as addition, subtraction, multiplication, and division. Notice that an expression does not have an equal sign. The value of the variable in each expression can change, and you can evaluate an expression using any value for the variable.

\[
50 - 2 \quad 4 - a \quad 12z \quad \frac{4x}{3}
\]

In the expressions above, \(a, z,\) and \(x\) are variables.

An expression that includes one or more variables is called an **algebraic expression**. You can use algebraic expressions to represent words or phrases. To help you solve word problems, you need to translate words or phrases into operations, variables, or expressions. Let’s start with addition and subtraction phrases. Take a look at this chart.
Table 6.1:

<table>
<thead>
<tr>
<th>Addition Phrases</th>
<th>Subtraction Phrases</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 plus $a$</td>
<td>4 less $d$</td>
</tr>
<tr>
<td>2 and $b$</td>
<td>6 less than $g$</td>
</tr>
<tr>
<td>4 more than $c$</td>
<td>$h$ fewer than 7</td>
</tr>
</tbody>
</table>

The bolded words in the phrases above tell you if you should use addition or subtraction, and the order of the terms. Read the word problem carefully to figure out which operation makes the most sense.

Let’s look at an example.

Abdul has $5 more than Xavier has. Write an algebraic expression to show the number of dollars Abdul has.

The phrase is “$5 more than Xavier.” Use a number, an operation sign, or a variable to represent each part of that phrase.

Let $x$ stand for the number of dollars Xavier has. “More than” means you are adding.

$$x + 5$$

So, the expression $x + 5$ represents the number of dollars Abdul has.

Let’s look at another example.

Change “6 less than a number” into an algebraic expression.

Let “a number” be the variable $x$. “Less than” means you are subtracting. Be careful about the order, 6 will follow the variable.

$$x - 6$$

The answer is $x - 6$.

Here is one more example.

Lian is $x$ inches shorter than Hannah. Hannah is 65 inches tall. Write an algebraic expression to show Lian’s height in inches.

The phrase is “$x$ inches shorter than Hannah.” You also know that Hannah’s height is 65 inches. “Shorter than” means you are subtracting. Be careful about the order, $x$ will follow 65.

$$65 - x$$

The answer is $65 - x$.

In a word problem, the word “is” means equals. When you see the word “is” you can set the expression equal to something. This is an equation.

Examples

Example 1

Earlier, you were given a problem about Sal’s biking team.
Sal needs to register his team of six people, but the maximum number of allowed racers on the course is 138 cyclists. Sal needs to write a single variable equation to figure out the maximum number of cyclists that can already be registered so that his team of six can join.

First, let $x$ be the maximum number of cyclists that can be registered. Six more than that will give you the maximum of 138 cyclists. When you can use the phrase “more than” you can use addition.

The answer is $x + 6 = 138$.

**Example 2**

Write an equation to represent the following phrase.

Four less than an unknown number is eighteen.

To figure this out, first let “an unknown number” be the variable $x$. Next, you know the operation is subtraction because of the key phrase “less than”.

So you can write $x - 4$ since the four is being taken away from the unknown number.

The word “is” means equals, and “eighteen” is 18.

$$x - 4 = 18$$

The answer is $x - 4 = 18$.

**Example 3**

Write an equation for the following phrase: a number plus five is ten.

Let $x$ be “a number”, “plus” means addition, and “is” means equals.

The answer is $x + 5 = 10$.

**Example 4**

Write an equation for the following phrase: six more than a number is eighteen.

Let $x$ be “a number”, “more than” means addition, and “is” means equals.

The answer is $x + 6 = 18$.

**Example 5**

Write an equation for the following phrase: fifteen less than a number is twenty.

Let $x$ be “a number”, “less than” means subtraction (but be careful about order), and “is” means equals.

The answer is $x - 15 = 20$.

**Review**

Write an expression for each phrase.

1. 5 more than a number
2. A number plus six
3. 8 and a number
4. Seven less than a number
5. Eight take away four
6. Nine more than a number

Write a simple equation for each phrase.

7. Five less than a number is ten.
8. Eight take away four is a number.
9. Five and a number is twelve.
10. Sixteen less than an unknown number is eighty.
11. Twenty and a number is fifty-five.
12. A number and fifteen is forty.
13. A number and twelve is sixty.
14. Fifteen less than a number is ninety.
15. Sixty less than a number is eighty.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.1.

Resources
6.4 Multiplication and Division Phrases as Equations

In this concept, you will learn how to write multiplication and division phrases as single variable equations.

There is a bunny population behind Ishmael’s house. Ishmael is helping the local park ranger track the population numbers. At the start of July there are 127 females. By the end of the month, Ishmael and the ranger count 415 new bunnies. What is the average number of bunnies that each female had during that time? How can Ishmael write an equation that shows this situation?

In this concept, you will learn how to write multiplication and division phrases as single variable equations.

**Writing Multiplication and Division Phrases as Equations**

Just as you can write addition and subtraction expressions from words or phrases, you can also write multiplication and division expressions. You can use key words to help you with this. The more familiar you become with the key words that identify a multiplication or division expression, the better you will become at writing expressions.

Here are some phrases that can be translated into multiplication or division expressions.

<table>
<thead>
<tr>
<th>Multiplication Phrases</th>
<th>Division Phrases</th>
</tr>
</thead>
<tbody>
<tr>
<td>9 times $k$</td>
<td>8 divided into $n$ groups</td>
</tr>
<tr>
<td>twice as much as $m$</td>
<td>$q$ shared equally by 3 people</td>
</tr>
<tr>
<td>$2 \times m$ or $2m$</td>
<td>$q \div 3$ or $\frac{q}{3}$</td>
</tr>
<tr>
<td>half of $r$</td>
<td>$r \div 2$ or $\frac{r}{2}$</td>
</tr>
<tr>
<td>one-third of $p$</td>
<td>$p \div 3$ or $\frac{p}{3}$</td>
</tr>
</tbody>
</table>

Remember, these words are only a helpful guide. You should always think about which operation makes sense for a
particular situation.

Let’s look at an example.

Write an algebraic expression to represent the phrase “three times a number, \( t \).”

Use a number, an operation sign, or a variable to represent each part of the phrase.

First, consider the phrase and look for key terms.

three times a number, \( t \)

Next, rewrite the phrase using numbers and operations.

\[ 3 \times t \]

Then, simplify the expression.

\[ 3t \]

The phrase is represented by the expression \( 3t \).

Here is another example.

Mr. Warren separated 30 students into \( n \) equal groups. Write an algebraic expression to represent the number of students in each group.

First, consider the phrase and look for key terms.

30 students into \( n \) equal groups

Separating 30 students into \( n \) equal groups means dividing 30 students into \( n \) equal groups.

Next, write a division expression.

\[ \frac{30}{n} \]

The phrase is represented by the expression \( \frac{30}{n} \).

When an expression equals something it is an equation. If the number of students in each group needed to be five, for instance, you could write the following equation.

\[ n = 5 \]

Then, you could evaluate the expression \( \frac{30}{n} \) using the given variable 5.

**Examples**

**Example 1**

Earlier, you were given a problem about Ishmael’s local bunnies.

There are 127 females and 415 bunnies. Ishmael needs to write an equation that represents the average number of bunnies each female had.

First, let \( n \) be the average number of bunnies per female.

\[ 127n \]

This is the total number of bunnies in the population.

Then, because you know there are 415 bunnies, you can write an equation.

\[ 127n = 415 \]
Example 2

Write a single variable equation for the following phrase.
Keith bought tickets to the movies. The tickets were $8.50 each. Keith spent a total of $34.00. How many tickets did Keith buy?
First, let $t$ represent the number of tickets that Keith bought.
The cost of each ticket is $8.50. The total price of the tickets will be the number of tickets, represented by $t$, times the price of each ticket.
Next, write an expression that represents this.
$8.5t$
Then, because you know the total cost of the tickets was $34.00, write an equation.
$8.5t = 34.00$
Write a multiplication or division equation for the following phrases.

Example 3

Four times a number is eight.
Let $x$ be “a number.”
“Times” means multiplication and “is” means equals.
The answer is $4x = 8$.

Example 4

Sixteen candles divided into a number of piles is two candles in each pile. Let $x$ be the number of piles.
“Divided into” means division but be careful about the order, and “is” means equals.
The answer is $\frac{16}{x} = 2$.

Example 5

The product of five and a number is fifteen.
Let $x$ be “a number.”
“Product” means you are multiplying, and “is” means equals.
The answer is $5x = 15$.

Review

Write an equation for each phrase.

1. The product of four and a number is twelve.
2. Six times a number is thirty.
3. Twelve times a number is forty-eight.
4. Fourteen times a number is twenty-eight.
5. The product of five and a number is thirty.
6. Eight times a number is sixty-four.
7. Twenty divided by a number is four.
8. Eighty divided by a number is four.
9. Nineteen times an unknown number is ninety-five.
10. Thirteen times an unknown number is thirty-nine.
11. Twelve divided into groups is six.
12. An unknown number divided by two is eight.
13. An unknown number divided by seven is fourteen.
14. An unknown number times five is thirty-five.
15. An unknown number divided by twelve is twelve.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.2.

Resources
6.5 Inequality Expressions

Here you’ll learn how to express inequalities in various forms, as well as how to graph inequalities on a number line. Suppose you’re having a party, and you know that the number of people attending will be greater than or equal to 25. How would you write this inequality? If you had to graph the solutions to this inequality on a number line, could you do it?

**Inequality Expressions**

An **algebraic inequality** is a mathematical sentence connecting an expression to a value, a variable, or another expression with an inequality sign.

Verbs that translate into inequalities are:

- > “greater than”
- ≥ “greater than or equal to”
- < “less than”
- ≤ “less than or equal to”
- ≠ “not equal to”

Solutions to one-variable inequalities can be graphed on a number line or in a coordinate plane.

Inequalities that “include” the value are shown as ≤ or ≥. The line underneath the inequality stands for “or equal to.” We show this relationship by coloring in the circle above this value on the number line. For inequalities without the “or equal to,” the circle above the value on the number line remains unfilled.

**Let’s graph the solutions to**

The inequality is asking for all real numbers larger than 3.

![Number Line](image)

**Now, let’s write the inequality pictured below:**

![Number Line](image)

The value of four is colored in, meaning that four is a solution to the inequality. The red arrow indicates values less than four. Therefore, the inequality is:

\[ x \leq 4 \]
Expressing Solutions to Inequalities

There are four ways to express solutions to inequalities:

1. Inequality notation: The answer is expressed as an algebraic inequality, such as \( d \leq \frac{1}{2} \).
2. Set notation: The inequality is rewritten using set notation brackets \{ \}. For example, \( \{ d | d \leq \frac{1}{2} \} \) is read, “The set of all values of \( d \), such that \( d \) is a real number less than or equal to one-half.”
3. Interval notation: This notation uses brackets to denote the range of values in an inequality.
   a. Square or “closed” brackets \([ \) indicate that the number is included in the solution.
   b. Round or “open” brackets \( ( \) indicate that the number is not included in the solution.

   Interval notation also uses the concept of infinity \( \infty \) and negative infinity \( -\infty \). For example, for all values of \( d \) that are less than or equal to \( \frac{1}{2} \), you could use set notation as follows: \( (-\infty, \frac{1}{2}] \).

4. As a graphed sentence on a number line.

Let’s describe the set of numbers contained by the given set notation for the following:

1. \((8, 24)\)

   \((8, 24)\) states that the solution is all numbers between 8 and 24 but does not include the numbers 8 and 24.

2. \([3, 12)\)

   \([3, 12)\) states that the solution is all numbers between 3 and 12, including 3 but not including 12.
6.5. Inequality Expressions

Examples

Example 1

Earlier, you were told to suppose that you’re having a party where the number of people attending will be greater than or equal to 25. How would you write this inequality? Could you graph it?

Remember that the symbol that translates to "greater than or equal to " is ≥. Let \( p \) represent the number of people attending the party. Then, the inequality that represents this situation is:

\[ p \geq 25 \]

The graph of this inequality would have a solid dot at 25 and have an arrow moving to the right since the number of people attending is greater than or equal to 25.

Example 2

Describe and graph the solution set expressed by (−∞, 3.25).

The solution set contains all numbers less than 3.25, not including 3.25.

The graph on the number line is:

![Number line graph](image)

Review

1. What are the four methods of writing the solution to an inequality?

Graph the solutions to the following inequalities using a number line.

2. \( x < -3 \)
3. \( x \geq 6 \)
4. \( x > 0 \)
5. \( x \leq 8 \)
6. \( x < -35 \)
7. \( x > -17 \)
8. \( x \geq 20 \)
9. \( x \leq 3 \)

Write the inequality that is represented by each graph.

10. [Graph 10]
11. [Graph 11]
12. [Graph 12]
Review (Answers)

To see the Review answers, open this PDF file and look for section 6.1.

Reason about and solve one-variable equations and inequalities.

6.EE.5 Understand solving an equation or inequality as a process of answering a question: Which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which $p$, $q$, and $x$ are all nonnegative rational numbers.

6.EE.8 Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.
Represent and analyze quantitative relationships between dependent and independent variables.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation \( d = 65t \) to represent the relationship between distance and time.
7.1 Function Rules for Input-Output Tables

In this concept, you will learn to evaluate and write function rules for an input-output table.

Geri is getting ready to go through the automatic car wash with her mom. Geri’s mom gives her several dollar bills and tells her to use it to get quarters for the car wash. Geri puts a dollar in the change machine and the machine gives her 4 quarters. She puts another dollar in the machine and it gives her 4 more quarters. Geri continues until she has 26 quarters. As she walks back to the car, she begins to think about what she put into the change machine and what came out.

This is a table to represent the change machine’s input and output.
### Table 7.1:

<table>
<thead>
<tr>
<th>Input (dollars)</th>
<th>Output (quarters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

What rule could Geri write to represent what happened to the input to equal the output?

In this concept, you will learn to evaluate and write function rules for an input-output table.

### Writing Function Rules for Input-Output Tables

A function is when one variable or term depends on another according to a rule. There is a special relationship between the two variables of the function where each value in the input applies to only one value in the output. These rules are called function rules, because they explain how the function operates. The function rule is the same thing as the expression. Here are some hints for writing function rules:

1. Decipher the pattern of the function by asking, “What happened to the input to get the output?”
2. Write the rule as an expression.

Take a look at the following function rule and determine if it is a rule for the data in the table below.

\[ x + 4 \]

First, substitute the input values in for \( x \) to see if you get the corresponding output value.

\[
\begin{align*}
2 + 4 &= 6 \\
3 + 4 &= 7 \\
4 + 4 &= 8 \\
5 + 4 &= 9
\end{align*}
\]

This does not equal the corresponding output value of 5.

Look at the other input values. Each term in the input became the term in the output when 3 was added to it. The rule states that four was added. Therefore, this is not a viable rule.

Here is another function.

\[ 5x \]

Determine if \( 5x \) is a function rule for the data in the table below.
First, substitute the input values in for $x$ to see if you get the corresponding output value.

\[
5x \\
5(20) \\
100
\]

Substitute another input value.

\[
5x \\
5(10) \\
50
\]

\[
5x \\
5(5) \\
25
\]

\[
5x \\
5(1) \\
5
\]

So, yes it is. In this case, each term in the input was multiplied by five to get the term in the output. Therefore this rule does work for this table.

**Examples**

**Example 1**

Earlier, you were given a problem about Geri and the change machine.

Geri knows that there are 4 quarters in a dollar, and that is why she put 6 dollars in the machine and received 24 quarters. How can Geri write this as a function rule?

This is a table to represent the change machine’s input and output.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>
First, look at the table and ask yourself, “What happened to \(x\) (input) to get \(y\) (output)?”

What happened to 5 to get 50? What happened to 6 to get 60 and so forth? If you look carefully, you will see that the input value \(x\) is multiplied by 10 to get the output value.

Next, use a variable for the input and write the rule.

You can write it as an expression, \(x \cdot 10\) or \(10x\). This is the function rule, \(10x\).

Then, see if the function rule \(10x\) works for each term in the table by plugging the input into the expression and seeing if it equals the listed output?

\[
\begin{align*}
10x & \\
10(5) & = 50 \\
10x & \\
10(6) & = 60 \\
10x & \\
10(7) & = 70 \\
10x & \\
10(8) & = 80
\end{align*}
\]

The answer is yes, this rule works for this table.

**Example 2**

Write a function rule to represent the data in this table.
First, look at the table and ask yourself, “What happened to $x$ (input) to get $y$ (output)?”

Here two operations were performed. The input value was multiplied by two and then one was subtracted.

Next, use a variable for the input and write the rule.

$2x - 1$

The answer or function rule is $2x - 1$.

Then, see if the function rule $2x - 1$ works for each term in the table by plugging the input into the expression and seeing if it equals the listed output.

Substitute the input values in for $x$ in the function $2x - 1$ to see if you get the results in the output column.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>19</td>
</tr>
</tbody>
</table>

$2x - 1$
$2(3) - 1$
$6 - 1$
$5$ (output)

$2x - 1$
$2(5) - 1$
$10 - 1$
$9$ (output)

$2x - 1$
$2(7) - 1$
$14 - 1$
$13$ (output)

$2x - 1$
$2(8) - 1$
$16 - 1$
$15$ (output)
2x − 1
2(10) − 1
20 − 1
19 (output)

The answer is correct.

Example 3

Determine whether the following rule makes sense for the input-output table.

Rule: 4x

Table 7.6:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
</tr>
</tbody>
</table>

First, let the input value be the variable x.
Next, substitute the input values in the expression for x.

\[ 4x = 10 \text{ (output)} \]
\[ 4(2) = 10 \]
\[ 8 \neq 10 \]

The answer is no, this rule does not work for this table.

Example 4

Determine whether the following rule makes sense for the input-output table.

Rule: 2x − 1

Table 7.7:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
</tr>
</tbody>
</table>

First, let the input value be the variable x.
Next, substitute the input values in the expression for x.
\[2x - 1 = 3 \text{ (output)}\]
\[2(2) - 1 = 3\]
\[4 - 1 = 3\]
\[3 = 3\]

\[2x - 1 = 5\]
\[2(3) - 1 = 5\]
\[6 - 1 = 5\]
\[5 = 5\]

\[2x - 1 = 7\]
\[2(4) - 1 = 7\]
\[8 - 1 = 7\]
\[7 = 7\]

\[2x - 1 = 11\]
\[2(6) - 1 = 11\]
\[12 - 1 = 11\]
\[11 = 11\]

The answer is yes, this rule does work for this table.

**Example 5**

Determine whether the following rule makes sense for the input-output table.

Rule: \(3x\)

**Table 7.8:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
</tr>
</tbody>
</table>

First, let the input value be the variable \(x\).

Next, substitute the input values in the expression for \(x\).

\[3x = 6 \text{ (output)}\]
\[3(2) = 6\]
\[6 = 6\]
$3x = 9$
$3(3) = 9$
$9 = 9$

$3x = 12$
$3(4) = 12$
$12 = 12$

$3x = 18$
$3(6) = 18$
$18 = 18$

The answer is yes, this rule does work for this table.

**Review**

Evaluate each given function rule to determine if the rule works for the data in the table.

1. $2x + 2$

<table>
<thead>
<tr>
<th>Table 7.9:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>

2. $3x$

<table>
<thead>
<tr>
<th>Table 7.10:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>

3. $5x + 1$

<table>
<thead>
<tr>
<th>Table 7.11:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>
### Table 7.11: (continued)

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
</tr>
</tbody>
</table>

4. \(2x\)

### Table 7.12:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

5. \(3x - 1\)

### Table 7.13:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
</tr>
</tbody>
</table>

6. \(2x + 1\)

### Table 7.14:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

7. \(4x\)

### Table 7.15:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

8. \(6x - 3\)
7.1. Function Rules for Input-Output Tables

**Table 7.16:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

9. $2x$

**Table 7.17:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

10. $3x - 3$

**Table 7.18:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

Create a table for each rule.

11. $5x$
12. $6x + 1$
13. $2x - 3$
14. $3x + 3$
15. $4x + 1$

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 12.11.

**Resources**
Timothy is planning a surprise birthday party for his sister at the local bowling alley. He has already arranged the food but wants to include bowling as an extra. The bowling alley has told Timothy that bowling shoes cost a flat rate of $2.00 and the cost to bowl is $3.00 per game. He has to figure out the cost for his sister and three friends to bowl and how many games they can bowl. Timothy only has $50.00 to spend on bowling. How can he figure out how many games they can bowl and have enough money to pay the total cost for four people?

In this concept, you will learn to evaluate function rules.

**Function Tables**

A function is a relation such that each member of the domain is paired with one and only one member of the range. A set of ordered pairs \((x, y)\) is a relation. The domain is the set of values made up of the \(x\)-coordinates of the ordered pairs while the range is the set of values made up of the \(y\)-values of the ordered pairs. A function table is an output/input table where the input value is a member of the domain and the output value is a member of the range. The output value is the result of an operation or operations performed on the input value and its value depends upon the input number.

Look at the following function table:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
</tr>
</tbody>
</table>
If ‘x’ represents the input value and ‘y’ represents the output value, the table can be used to write a function rule. A function rule is an expression written in either words or symbols to represent the operation or operations performed on the input number to give the output number. From the above table it is obvious that each output number ‘y’ is the result of doubling the input number ‘x.’ The function rule written using symbols is the equation:

\[ y = 2x \]

Let’s apply a function rule to complete the following input/output table.

Use the function rule \( y = 3x + 2 \) to complete the table below.

**Table 7.20:**

<table>
<thead>
<tr>
<th>Input (x)</th>
<th>Output (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
</tr>
</tbody>
</table>

First, substitute the input value of 2 into the function rule for the variable ‘x’.

\[ y = 3x + 2 \]
\[ y = 3(2) + 2 \]

Next, perform the multiplication to clear the parenthesis.

\[ y = 6 + 2 \]
\[ y = 8 \]

The answer is 8.

The output value is 8 when the input value is 2.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.

\[ x = 3 \]
\[ y = 3x + 2 \]
\[ y = 3(3) + 2 \]
\[ y = 9 + 2 \]
\[ y = 11 \]

The answer is 11.
7.2. Evaluating Function Rules

\[
x = 4 \\
y = 3x + 2 \\
y = 3(4) + 2 \\
y = 12 + 2 \\
y = 14 \\
\]

The answer is 14.

\[
x = 5 \\
y = 3x + 2 \\
y = 3(5) + 2 \\
y = 15 + 2 \\
y = 17 \\
\]

The answer is 17.

Then, complete the table by filling in the calculated output numbers.

**Examples**

**Example 1**

Earlier, you were given a problem about Timothy and the surprise birthday party. He needs to figure out how many games of bowling his sister and her friends can play for $50.00 or less. First, he must write a function rule to represent the information he has from the bowling alley. Shoes are a flat rate of $2.00 and each game played costs $3.00. The function rule for the information is: \( y = 3x + 2 \) where \( y \) is the total cost and \( x \) is the number of games played.

Next, create an input/output table.

**Table 7.21:**

<table>
<thead>
<tr>
<th>Input ((x))</th>
<th>Output ((y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5.00</td>
</tr>
<tr>
<td>2</td>
<td>$8.00</td>
</tr>
<tr>
<td>3</td>
<td>$11.00</td>
</tr>
<tr>
<td>4</td>
<td>$14.00</td>
</tr>
</tbody>
</table>

First, substitute the input value of 1 into the function rule for the variable \( x \).

\[
y = 3x + 2 \\
y = 3(1) + 2 \\
\]

Next, perform the multiplication to clear the parenthesis.

\[
y = 3(1) + 2 \\
y = 3 + 2 \\
\]
Next, perform the addition on the right side of the equation.

\[
y = 3 + 2
\]
\[
y = 5
\]

The answer is 5.
The output value is $5.00 for one person to bowl one game.
Next, write the output number in the table.
Then, multiply $5.00 by 4 to determine the cost for four people to bowl one game.

\[
$5.00 \times 4 = $20.00
\]
The answer is $20.00.
It will cost $20.00 for his sister and three friends to bowl one game.
First, substitute the input value of 2 into the function rule for the variable ‘x.’

\[
y = 3x + 2
\]
\[
y = 3(2) + 2
\]

Next, perform the multiplication to clear the parenthesis.

\[
y = 3(2) + 2
\]
\[
y = 6 + 2
\]

Next, perform the addition on the right side of the equation.

\[
y = 6 + 2
\]
\[
y = 8
\]

The answer is 8.
The output value is $8.00 for one person to bowl two games.
Next, write the output number in the table.
Then, multiply $8.00 by 4 to determine the cost for four people to bowl two games.

\[
$8.00 \times 4 = $32.00
\]
The answer is $32.00.
It will cost $32.00 for his sister and three friends to bowl two games.
First, substitute the input value of 3 into the function rule for the variable ‘x.’
\[ y = 3x + 2 \]
\[ y = 3(3) + 2 \]

Next, perform the multiplication to clear the parenthesis.

\[ y = 3(3) + 2 \]
\[ y = 9 + 2 \]

Next, perform the addition on the right side of the equation.

\[ y = 9 + 2 \]
\[ y = 11 \]

The answer is 11.

The output value is $11.00 for one person to bowl three games.

Next, write the output number in the table.

Then, multiply $11.00 by 4 to determine the cost for four people to bowl three games.

\[ $11.00 \times 4 = $44.00 \]

The answer is $44.00.

It will cost $44.00 for his sister and three friends to bowl three games.

First, substitute the input value of 4 into the function rule for the variable 'x.'

\[ y = 3x + 2 \]
\[ y = 3(4) + 2 \]

Next, perform the multiplication to clear the parenthesis.

\[ y = 3(4) + 2 \]
\[ y = 12 + 2 \]

Next, perform the addition on the right side of the equation.

\[ y = 12 + 2 \]
\[ y = 14 \]

The answer is 14.

The output value is $14.00 for one person to bowl four games.

Next, write the output number in the table.
Then, multiply $14.00 by 4 to determine the cost for four people to bowl four games.

\[ \$14.00 \times 4 = \$56.00 \]

The answer is $56.00.

It will cost $56.00 for his sister and three friends to bowl four games.

Timothy has enough money for his sister and three friends to bowl three games.

**Examples 2**

Use the following function rule to complete the input/output table.

\[ y = 2x - 5 \]

**Table 7.22:**

<table>
<thead>
<tr>
<th>Input ((x))</th>
<th>Output ((y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>-15</td>
</tr>
<tr>
<td>-1</td>
<td>-7</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

First, substitute the input value of -5 into the function rule for the variable \(x\).

\[
\begin{align*}
  y &= 2x - 5 \\
  y &= 2(-5) - 5 \\
  y &= -10 - 5 \\
  y &= -15
\end{align*}
\]

The answer is -15.

The output value is -15 when the input value is -5.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.
7.2. Evaluating Function Rules

\[ \begin{align*}
    x &= -1 \\
    y &= 2x - 5 \\
    &= 2(-1) - 5 \\
    &= -2 - 5 \\
    &= -7
\end{align*} \]

The answer is -7.

\[ \begin{align*}
    x &= 2 \\
    y &= 2x - 5 \\
    &= 2(2) - 5 \\
    &= 4 - 5 \\
    &= -1
\end{align*} \]

The answer is -1.

\[ \begin{align*}
    x &= 6 \\
    y &= 2x - 5 \\
    &= 2(6) - 5 \\
    &= 12 - 5 \\
    &= 7
\end{align*} \]

The answer is 7.

**Example 3**

Use the given function rule to complete the input/output table:

\[ y = 4x - 3 \]

<table>
<thead>
<tr>
<th>Input (x)</th>
<th>Output (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
</tr>
<tr>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>9</td>
<td>33</td>
</tr>
</tbody>
</table>

First, substitute the input value of 4 into the function rule for the variable ’x.’

\[ \begin{align*}
    y &= 4x - 3 \\
    &= 4(4) - 3
\end{align*} \]

Next, perform the multiplication to clear the parenthesis.
Next, perform the subtraction on the right side of the equation.

\[ y = 16 - 3 \]
\[ y = 13 \]

The answer is 13.

The output value is 13 when the input value is 4.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.

\[ x = 5 \]
\[ y = 4x - 3 \]
\[ y = 4(5) - 3 \]
\[ y = 20 - 3 \]
\[ y = 17 \]

The answer is 17.

\[ x = 7 \]
\[ y = 4x - 3 \]
\[ y = 4(7) - 3 \]
\[ y = 28 - 3 \]
\[ y = 25 \]

The answer is 25.

\[ x = 9 \]
\[ y = 4x - 3 \]
\[ y = 4(9) - 3 \]
\[ y = 36 - 3 \]
\[ y = 33 \]

The answer is 33.

**Example 4**

Use the given function rule to complete the input/output table:

\[ y = -2x + 7 \]
First, substitute the input value of -14 into the function rule for the variable \( x \).\[
\begin{align*}
y &= -2x + 7 \\
y &= -2(-14) + 7
\end{align*}
\]
Next, perform the multiplication to clear the parenthesis.
\[
\begin{align*}
y &= -2(-14) + 7 \\
y &= 28 + 7
\end{align*}
\]
Next, perform the addition on the right side of the equation.
\[
\begin{align*}
y &= 28 + 7 \\
y &= 35
\end{align*}
\]
The answer is 35.
The output value is 35 when the input value is -14.

Then, write the output number in the table.
Repeat the above process for each of the given input numbers.
\[
\begin{align*}
x &= -9 \\
y &= -2x + 7 \\
y &= -2(-9) + 7 \\
y &= 18 + 7 \\
y &= 25
\end{align*}
\]
The answer is 25.
\[
\begin{align*}
x &= 4 \\
y &= -2x + 7 \\
y &= -2(4) + 7 \\
y &= -8 + 7 \\
y &= -1
\end{align*}
\]
The answer is -1.
\[x = 11\]
\[y = -2x + 7\]
\[y = -2(11) + 7\]
\[y = -22 + 7\]
\[y = -15\]

The answer is -15.

**Example 5**

Use the given function rule to complete the input/output table:

\[y = 3x - 14\]

**Table 7.25:**

<table>
<thead>
<tr>
<th>Input (x)</th>
<th>Output (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>-44</td>
</tr>
<tr>
<td>-7</td>
<td>-35</td>
</tr>
<tr>
<td>2</td>
<td>-8</td>
</tr>
<tr>
<td>9</td>
<td>13</td>
</tr>
</tbody>
</table>

First, substitute the input value of -10 into the function rule for the variable \(x\).

\[y = 3x - 14\]
\[y = 3(-10) - 14\]

Next, perform the multiplication to clear the parenthesis.

\[y = 3(-10) - 14\]
\[y = -30 - 14\]

Next, perform the addition on the right side of the equation.

\[y = -30 - 14\]
\[y = -44\]

The answer is -44.

The output value is -44 when the input value is -10.

Then, write the output number in the table.

Repeat the above process for each of the given input numbers.
7.2. Evaluating Function Rules

\[ x = -7 \]
\[ y = 3x - 14 \]
\[ y = 3(-7) - 14 \]
\[ y = -21 - 14 \]
\[ y = -35 \]

The answer is -35.

\[ x = 2 \]
\[ y = 3x - 14 \]
\[ y = 3(2) - 14 \]
\[ y = 6 - 14 \]
\[ y = -8 \]

The answer is -8.

\[ x = 9 \]
\[ y = 3x - 14 \]
\[ y = 3(9) - 14 \]
\[ y = 27 - 14 \]
\[ y = 13 \]

The answer is 13.

**Review**

For numbers 1-5, find each output if the function rule is \( y = 3x + 2 \).

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>

For numbers 6-10, find each output if the function rule is \( y = 4x \).

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
For numbers 11-15, find each output if the function rule is \( y = -3x \).

**Table 7.28:**

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>12.</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>14.</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>15.</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Answer each question about functions.

16. A pastry chef needs to purchase enough dough for her cookies. She buys one pound of dough for every twenty cookies she is going to make. She uses the function \( d(c) = \frac{c}{20} \) where \( c \) is the number of cookies and \( d \) is the pounds of dough she should buy. Identify which variable is the domain and which is the range.

17. Evaluate the function \( f(x) = 2x + 7 \) when the domain is \( \{-3, -1, 1, 3\} \).

18. Evaluate the function \( f(x) = \frac{2}{3}x - 6 \) when the domain is \( \{-10, -5, 0, 5, 10\} \).

19. Evaluate the function \( f(x) = 3x - 1 \) when the domain is \( \{5, 6, 7, 8, 9\} \).

20. Evaluate the function \( f(x) = x - 9 \) when the domain is \( \{1, 2, 3, 4, 5\} \).

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 9.2.

Represent and analyze quantitative relationships between dependent and independent variables.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation \( d = 65t \) to represent the relationship between distance and time.
Module 5: Area, Surface Area, and Volume Problems

Chapter Outline

8.1 Area and Perimeter of Triangles
8.2 Area of Squares and Rectangles
8.3 Area and Perimeter of Trapezoids
8.4 Area and Perimeter of Rhombuses and Kites
8.5 Triangle Area
8.6 Area of a Parallelogram
8.7 Area of Composite Shapes Involving Triangles
8.8 Volume of Prisms Using Unit Cubes
8.9 Volume of Rectangular Prisms
8.10 Polygon Classification in the Coordinate Plane
8.11 Cross-Sections and Nets

Solve real-world and mathematical problems involving area, surface area, and volume.

6.G.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas V = l w h and V = b h to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

6.G.3 Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

6.G.4 Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Review, connect, apply these earlier standards mentioned earlier in the Flexbook:

Apply and extend previous understandings of arithmetic to algebraic expressions.

6.EE.2 Write, read, and evaluate expressions in which letters stand for numbers.

c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas V = s^3 and A = 6 s^2 to find the volume and surface area of a cube with sides of length s = 1/2.
Reason about and solve one-variable equations and inequalities.

6.EE.5 Understand solving an equation or inequality as a process of answering a question: Which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which $p$, $q$, and $x$ are all nonnegative rational numbers.
Area and Perimeter of Triangles

Here you’ll learn how to find the area and perimeter of a triangle given its base and height.

Area and Perimeter of Triangles

The formula for the area of a triangle is half the area of a parallelogram.

\[ A = \frac{1}{2}bh \]

What if you were given a triangle and the size of its base and height? How could you find the total distance around the triangle and the amount of space it takes up?

Area of a Triangle: \( A = \frac{1}{2}bh \) or \( A = \frac{bh}{2} \).

Examples

For Examples 1 and 2, use the following triangle.
Example 1

Find the height of the triangle.
Use the Pythagorean Theorem to find the height.

\[ 8^2 + h^2 = 17^2 \]
\[ h^2 = 225 \]
\[ h = 15 \text{ in} \]

Example 2

Find the perimeter.
We need to find the hypotenuse. Use the Pythagorean Theorem again.

\[ (8 + 24)^2 + 15^2 = h^2 \]
\[ h^2 = 1249 \]
\[ h \approx 35.3 \text{ in} \]

The perimeter is \( 24 + 35.3 + 17 \approx 76.3 \text{ in} \).

Example 3

Find the area of the triangle.

To find the area, we need to find the height of the triangle. We are given two sides of the small right triangle, where the hypotenuse is also the short side of the obtuse triangle.
8.1. Area and Perimeter of Triangles

\[
3^2 + h^2 = 5^2 \\
9 + h^2 = 25 \\
h^2 = 16 \\
h = 4 \\
A = \frac{1}{2}(4)(7) = 14 \text{ units}^2
\]

**Example 4**

Find the perimeter of the triangle in Example 3.

To find the perimeter, we need to find the longest side of the obtuse triangle. If we used the black lines in the picture, we would see that the longest side is also the hypotenuse of the right triangle with legs 4 and 10.

\[
4^2 + 10^2 = c^2 \\
16 + 100 = c^2 \\
c = \sqrt{116} \approx 10.77
\]

The perimeter is \(7 + 5 + 10.77 \approx 22.77 \text{ units}\)

**Example 5**

Find the area of a triangle with base of length 28 cm and height of 15 cm.

The area is \(\frac{1}{2}(28)(15) = 210 \text{ cm}^2\).

**Review**

Use the triangle to answer the following questions.

1. Find the height of the triangle by using the geometric mean.
2. Find the perimeter.
3. Find the area.

Find the area of the following shape.

5. What is the height of a triangle with area 144 $m^2$ and a base of 24 m?

In questions 6-11 we are going to derive a formula for the area of an equilateral triangle.

6. What kind of triangle is $\triangle ABD$? Find $AD$ and $BD$.
7. Find the area of $\triangle ABC$.
8. If each side is $x$, what is $AD$ and $BD$?
9. If each side is $x$, find the area of $\triangle ABC$.
10. Using your formula from #9, find the area of an equilateral triangle with 12 inch sides.
11. Using your formula from #9, find the area of an equilateral triangle with 5 inch sides.

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 10.3.
8.2 Area of Squares and Rectangles

In this concept, you will learn how to find the area of squares and rectangles.

Let’s Think About It

Allen bought a new house and is planning to replace the carpet in two of the bedrooms. One bedroom is a shaped like a square and the other is rectangular. He needs to figure out how much carpet is needed to cover both bedroom floors.

The square bedroom has side lengths of 9 feet.

The rectangular bedroom has a length of 12 feet and a width of 8 feet.

In order for Allen to figure out how much carpet he needs to buy for these two bedrooms, he needs to find the area of each bedroom.

In this concept, you will learn how to find the area of squares and rectangles.

Guidance

The inside space of a figure is known as the area of the figure. Carpeting a floor, grass on the ground, or anything else covering the space inside of a figure are all examples of area.

There is a formula for finding the area of different shapes. In this concept, this concept will show you how to find the area of both squares and rectangles.

This is the formula used to find the area of a square:
Remember, the dot is another symbol for multiplication. To figure out the area of the square, multiply one side times the other side. Since all sides in a square are equal, the two numbers will be the same. The unit of measurement is kept in the equation to help you remember what the correct label should be.

Here is how to use the formula for finding the area of a square:

\[ A = \text{side} \times \text{side} \]

Now, think of the equation as multiplying two different things; the numbers and the units of measurement.

\[ A = 6 \text{ ft} \times 6 \text{ ft} \]

Solve both equations.

\[ A = 6 \times 6 = 36 \]
\[ A = \text{ft} \times \text{ft} = \text{ft}^2 \]

Think back to what you know about exponents. When you multiply two of the same thing together, you can write it in exponential form. Another way to label area is to write square feet (sq. ft.).

The answer is 36 sq. ft. or 36 ft²

This is the formula used to find the area of a rectangle:

\[ A = \text{length} \times \text{width} \]

To find the area of a rectangle, use the measurements for length and width instead of side length. These numbers may be different from each other because not all rectangles have the same side lengths like squares do.
This rectangle measures a length of 5 meters and a width of 3 meters. Just like the area formula for a square, multiply those two numbers to find the area of the rectangle. Remember, two variables next to each other in an equation indicate multiplication, just like the dot symbol or traditional multiplication symbol. The symbols will be used interchangeably.

To find the area of a rectangle, multiply length times width.

\[ A = 5m \cdot 3m \]
\[ A = 5 \times 3 \]
\[ A = \text{meters} \times \text{meters} \]

The formula shows 5 meters times 3 meters. Multiply the measurement part \((5 \times 3)\), then multiply the units of measurement.

The answer is 15 sq. m or 15 \(m^2\).

The label for area of a rectangle is similar to the label for area of a square from above. The label can be written in exponential form or written in abbreviation.

**Guided Practice**

Find the area of a rectangle with a length of 7 ft. and a width of 4 ft.

First, write out the equation

\[ A = lw \]
\[ A = 7ft \cdot 4ft \]

Next, multiply the two numbers.

\[ A = 7 \cdot 4 = 28 \]

Then, multiply the units of measurement.

\[ A = ft \cdot ft = ft^2 \]
Now, put your answer together to show the final solution.

\[ A = 28 \text{ ft}^2 \]

The answer is 28 sq. ft.

Once you do more examples you may find yourself skipping the step of multiplying the units of measurement. This may become natural to square the unit of measurement for any area problem.

**Examples**

**Example 1**

Find the area of a square with side lengths of 10 inches.
First, write your equation.

\[ A = s \cdot s \]
\[ A = 10 \text{in} \cdot 10 \text{in} \]

Next, solve the equation by multiplying the two side lengths.

\[ A = 10 \cdot 10 = 100 \]

Then, write your final solution. Remember to write the squared unit of measurement after the numerical answer.

\[ A = 100 \text{in}^2 \]

The answer is 100 sq. in.

**Example 2**

Find the area of a rectangle with a length of 6 feet and a width of 5 feet.
First, write your equation.

\[ A = lw \]
\[ A = 6 \text{ft} \cdot 5 \text{ft} \]

Next, solve the equation by multiplying the length times the width.

\[ A = 6 \cdot 5 = 30 \]
8.2. Area of Squares and Rectangles

Then, write your final solution. Remember to write the squared unit of measurement after the numerical answer.

\[ A = 30 \text{ ft}^2 \]

The answer is 30 sq. ft.

**Example 3**

Find the area of a rectangle with a length of 9 meters and a width of 8 meters.

First, write your equation.

\[ A = lw \]
\[ A = 9 \text{m} \cdot 8 \text{m} \]

Next, solve the equation by multiplying the length times the width.

\[ A = 9 \cdot 8 = 72 \]

Then, write your final solution. Remember to write the squared unit of measurement after the numerical answer.

\[ A = 72 \text{m}^2 \]

The answer is 72 sq. m

**Follow Up**

Remember Allen and his new house that he wants to re-carpet?
The dimensions of Allen’s two bedrooms are given in the beginning of the concept. His square bedroom has side lengths of 9 feet. The other room is a rectangle with dimensions of 12 feet by 8 feet. To begin solving this problem, find the area of each of the bedrooms.

The square bedroom has a side length of 9 feet.

\[ A = s \cdot s \]
\[ A = 9 \times 9 = 81 \text{ sq. feet} \]

The square bedroom has an area of 81 square feet.

The rectangular bedroom has a length of 12 feet and width of 8 feet.

\[ A = 12 \text{ ft} \cdot 8 \text{ ft} \]
\[ A = 12 \times 8 \]
\[ A = \text{feet} \times \text{feet} \]

The rectangular bedroom has an area of 96 square feet.

Allen wants the total area of his two bedrooms to know how much carpet to buy, so add the two areas together.

\[ 81 + 96 = 177 \text{ square feet} \]

The answer is 177 square feet.

**Video Review**

Click image to the left or use the URL below.
**URL:** [https://www.ck12.org/flx/render/embeddedobject/5309](https://www.ck12.org/flx/render/embeddedobject/5309)

Click image to the left or use the URL below.
**URL:** [https://www.ck12.org/flx/render/embeddedobject/5310](https://www.ck12.org/flx/render/embeddedobject/5310)

**Explore More**

Find the area of each of the following figures. Make sure you label your answer correctly.
1. A square with a side length of 6 inches.
2. A square with a side length of 4 inches.
3. A square with a side length of 8 centimeters.
4. A square with a side length of 12 centimeters.
5. A square with a side length of 9 meters.
6. A rectangle with a length of 6 inches and a width of 4 inches.
7. A rectangle with a length of 9 meters and a width of 3 meters.
8. A rectangle with a length of 4 meters and a width of 2 meters.
9. A rectangle with a length of 17 feet and a width of 12 feet.
10. A rectangle with a length of 22 feet and a width of 18 feet.
11. A square with a side length of 13 feet.
12. A square with a side length of 18 feet.
13. A square with a side length of 21 feet.
14. A rectangle with a length of 18 feet and a width of 13 feet.
15. A rectangle with a length of 60 feet and a width of 27 feet.
16. A rectangle with a length of 57 feet and a width of 22 feet.

**Answers for Explore More Problems**

To view the Explore More answers, open this PDF file and look for section 2.5.
8.3 Area and Perimeter of Trapezoids

Here you’ll learn how to find the area and perimeter of a trapezoid given its two bases and its height.

Area and Perimeter of Trapezoids

A trapezoid is a quadrilateral with one pair of parallel sides. The parallel sides are called the bases and we will refer to the lengths of the bases as $b_1$ and $b_2$. The perpendicular distance between the parallel sides is the height of the trapezoid. The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$ where $h$ is always perpendicular to the bases.

What if you were given a trapezoid and the size of its two bases as well as its height? How could you find the total distance around the trapezoid and the amount of space it takes up?

Examples

Example 1

Find the area of the trapezoid.

![Diagram of a trapezoid with bases 21 and 18, and height 41]
Use the formula for the area of a trapezoid.
\[
\frac{1}{2}(18)(41 + 21) = 558 \text{ units}^2
\]

**Example 2**

Find the area of the trapezoid. *Round your answers to the nearest hundredth.*

Use the formula for the area of a trapezoid.
\[
\frac{1}{2}(5)(16 + 9) = 62.5 \text{ units}^2
\]

**Example 3**

Find the area of the trapezoid.

\[
A = \frac{1}{2}(11)(14 + 8)
A = \frac{1}{2}(11)(22)
A = 121 \text{ units}^2
\]

**Example 4**

Find the area of the trapezoid.
Example 5

Find the perimeter and area of the trapezoid.

Even though we are not told the length of the second base, we can find it using special right triangles. Both triangles at the ends of this trapezoid are isosceles right triangles, so the hypotenuses are $4\sqrt{2}$ and the other legs are of length 4.

\[
\begin{align*}
P &= 8 + 4\sqrt{2} + 16 + 4\sqrt{2} \\
A &= \frac{1}{2}(4)(8 + 16)
\end{align*}
\]

\[
\begin{align*}
P &= 24 + 8\sqrt{2} \approx 35.3 \ units \\
A &= 48 \ units^2
\end{align*}
\]

Review

Find the area and perimeter of the following shapes. *Round your answers to the nearest hundredth.*

1. Trapezoid with bases 3 in and 7 in and height of 3 in.
2. Trapezoid with bases 6 in and 8 in and height of 5 in.
3. Trapezoid with bases 10 in and 26 in and height of 2 in.
4. Trapezoid with bases 15 in and 12 in and height of 10 in.
5. Trapezoid with bases 4 in and 23 in and height of 21 in.
8. Trapezoid with bases 9 in and 4 in and height of 1 in.
9. Trapezoid with bases 12 in and 8 in and height of 16 in.
10. Trapezoid with bases 26 in and 14 in and height of 19 in.

Use the given figures to answer the questions.

11. What is the perimeter of the trapezoid?
12. What is the area of the trapezoid?

13. What is the perimeter of the trapezoid?
14. What is the area of the trapezoid?
15. What is the perimeter of the trapezoid?
16. What is the area of the trapezoid?

**Review (Answers)**

To see the Review answers, open this [PDF file](#) and look for section 10.5.
Here you’ll learn how to find the area and perimeter of a kite or a rhombus given its two diagonals.

**Area and Perimeter of Rhombuses and Kites**

Recall that a **rhombus** is a quadrilateral with four congruent sides and a **kite** is a quadrilateral with distinct adjacent congruent sides. Both of these quadrilaterals have perpendicular diagonals, which is how we are going to find their areas.

Notice that the diagonals divide each quadrilateral into 4 triangles. If we move the two triangles on the bottom of each quadrilateral so that they match up with the triangles above the horizontal diagonal, we would have two rectangles.

So, the height of these rectangles is half of one of the diagonals and the base is the length of the other diagonal.

The **area of a rhombus or a kite** is \( A = \frac{1}{2} d_1 d_2 \)

What if you were given a kite or a rhombus and the size of its two diagonals? How could you find the total distance around the kite or rhombus and the amount of space it takes up?
Examples

Example 1

Find the perimeter and area of the kite below.

![Kite Diagram]

In a kite, there are two pairs of congruent triangles. Use the Pythagorean Theorem to find the lengths of sides or diagonals.

\[
\begin{align*}
\text{Smaller diagonal portion} & : \\
20^2 + d_s^2 &= 25^2 \\
d_s^2 &= 225 \\
d_s &= 15 \text{ units} \\
\text{Larger diagonal portion} & : \\
20^2 + d_l^2 &= 35^2 \\
d_l^2 &= 825 \\
d_l &= 5 \sqrt{33} \text{ units}
\end{align*}
\]

\[
A = \frac{1}{2} \left( 15 + 5 \sqrt{33} \right) (40) \approx 874.5 \text{ units}^2 \\
P = 2(25) + 2(35) = 120 \text{ units}
\]

Example 2

Find the area of a rhombus with diagonals of 6 in and 8 in.

The area is \( \frac{1}{2}(8)(6) = 24 \text{ in}^2 \).

Example 3

Find the perimeter and area of the rhombus below.
8.4. Area and Perimeter of Rhombuses and Kites

In a rhombus, all four triangles created by the diagonals are congruent.

To find the perimeter, you must find the length of each side, which would be the hypotenuse of one of the four triangles. Use the Pythagorean Theorem.

\[12^2 + 8^2 = side^2\]
\[144 + 64 = side^2\]
\[side = \sqrt{208} = 4\sqrt{13}\]

\[P = 4 \left( 4\sqrt{13} \right) = 16\sqrt{13} \text{ units}\]

**Example 4**

Find the perimeter and area of the rhombus below.

In a rhombus, all four triangles created by the diagonals are congruent.

Here, each triangle is a 30-60-90 triangle with a hypotenuse of 14. From the special right triangle ratios the short leg is 7 and the long leg is \(7\sqrt{3}\).

\[P = 4 \cdot 14 = 56 \text{ units}\]
\[A = \frac{1}{2} \cdot 14 \cdot 14\sqrt{3} = 98\sqrt{3} \text{ units}^2\]

**Example 5**

The vertices of a quadrilateral are \(A(2,8), B(7,9), C(11,2), \) and \(D(3,3)\). Show \(ABCD\) is a kite and find its area.

After plotting the points, it looks like a kite. \(AB = AD\) and \(BC = DC\). The diagonals are perpendicular if the slopes are negative reciprocals of each other.
The diagonals are perpendicular, so \(ABCD\) is a kite. To find the area, we need to find the length of the diagonals, \(AC\) and \(BD\).

\[
\begin{align*}
   d_1 &= \sqrt{(2 - 11)^2 + (8 - 2)^2} \\
   &= \sqrt{(-9)^2 + 6^2} \\
   &= \sqrt{81 + 36} = \sqrt{117} = 3\sqrt{13} \\
\end{align*}
\]

\[
\begin{align*}
   d_2 &= \sqrt{(7 - 3)^2 + (9 - 3)^2} \\
   &= \sqrt{4^2 + 6^2} \\
   &= \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13} \\
\end{align*}
\]

Plug these lengths into the area formula for a kite. \(A = \frac{1}{2} \left( 3\sqrt{13} \right) \left( 2\sqrt{13} \right) = 39 \text{ units}^2\)

**Review**

1. Do you think all rhombi and kites with the same diagonal lengths have the same area? Explain your answer.

Find the area of the following shapes. **Round your answers to the nearest hundredth.**

2. \[
\begin{align*}
   16 & \\
   20 & \\
\end{align*}
\]

3. \[
\begin{align*}
   8 & \\
   6 & \\
\end{align*}
\]
Find the area and perimeter of the following shapes. *Round your answers to the nearest hundredth.*
For Questions 12 and 13, the area of a rhombus is $32 \text{ units}^2$.

12. What would the product of the diagonals have to be for the area to be $32 \text{ units}^2$?
13. List two possibilities for the length of the diagonals, based on your answer from #12.

For Questions 14 and 15, the area of a kite is $54 \text{ units}^2$.

14. What would the product of the diagonals have to be for the area to be $54 \text{ units}^2$?
15. List two possibilities for the length of the diagonals, based on your answer from #14.

Sherry designed the logo for a new company, made up of 3 congruent kites.

16. What are the lengths of the diagonals for one kite?
17. Find the area of one kite.
18. Find the area of the entire logo.

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 10.6.
Terry has two tickets to his favorite college football team’s next home game, so he is planning to take his younger brother with him to the game. Terry wants to make two pennants for the game, so he buys a piece of material shaped like a rectangle with a base of 18 inches and a height of 12 inches. Terry plans to divide the material into two triangles. How can Terry figure out the area of each of the triangles that will be cut from the material for his pennants?

In this concept, you will learn to find the areas of triangles given a base and a height.

Finding the Area of a Triangle

Triangles are not parallelograms because they only have three sides, but they are related to parallelograms. Take a look the parallelogram below and see if you can figure out the connection.
If you look carefully, you will notice that the parallelogram can be divided into two triangles.

A rectangle is a type of parallelogram with four right angles. You can divide a rectangle into two triangles also.

Notice that a rectangle is divided into two right triangles, triangles containing a right angle.

A square is a type of rectangle with four equal sides and four right angles. A square can also be divided into two right triangles.
If a parallelogram can be divided into two triangles, then the area of a triangle is one-half the area of a parallelogram. Let’s look at how this works.

What is the area of this parallelogram?

To find the area of the parallelogram, multiply the base by the height.

\[
A = bh
\]
\[
A = 2(5)
\]
\[
A = 10 \text{ sq. inches.}
\]
A parallelogram can be divided into two triangles.

If you divide the area of the parallelogram in half, that will give you the area of one of the triangles.

\[ 10 \div 2 = 5 \text{ sq. inches} \]

Based on this information, you can write the following formula for finding the area of a triangle.

\[ A = \frac{1}{2}bh \]

A triangle is one-half of a parallelogram, so the formula for the parallelogram multiplied by one-half is the formula for finding the area of a triangle.

**Examples**

**Example 1**

Earlier, you were given a problem about Terry and his sports pennants.

Terry has always been a Gator football fan, so he is excited about attending one of the team’s home games. He has purchased material to make two pennants for the game. The material is shaped like a rectangle, and Terry plans to divide it into two triangles for the pennants. The material has a 38-inch base and a height of 30 inches. Terry wants to know the size of the two triangles after he cuts the material. How can Terry figure out the area of each of the triangles that will form his pennants?

Here is a look at the piece of material.
First, recall that the area of a rectangle is base times height and a triangle is one half the area of a rectangle.

\[
\text{Base} \times \text{Height} = \text{Area of a Rectangle} \\
18 \times 12 = 216 \\
= 216 \text{ square inches}
\]

Next, substitute the area value for the rectangle (or the base \( \times \) height) in the triangle area formula.

\[
A = \frac{1}{2}bh \\
A = \frac{1}{2}(216)
\]

Then, solve the equation.

\[
A = \frac{1}{2}(216) \\
A = 108 \text{ square inches}
\]

The answer is each of the triangles will have an area of 108 square inches.

**Example 2**

Find the area of the triangle with a base of 6 feet and a height of 5 feet.

First, substitute the values for the base and height into the triangle area formula.

\[
A = \frac{1}{2}bh \\
A = \frac{1}{2}(6)(5)
\]

Next, solve the equation.

\[
A = \frac{1}{2}(30) \\
A = 15 \text{ square feet}
\]

The answer is 15 square feet.

**Example 3**

Using the area of the following parallelogram, find the area of one of the triangles inside the parallelogram.

Area of a rectangle is 12 sq. inches

First, recall that the area of a rectangle is base times height and a triangle is one half the area of a rectangle.

Next, substitute the area value for the base and height in the triangle area formula.

\[
A = \frac{1}{2}bh \\
A = \frac{1}{2}(12)
\]
Then, solve the equation.

\[ A = \frac{1}{2}(12) \]
\[ A = 6 \text{ square inches} \]

The answer is 6 square inches.

**Example 4**

Using the area of the following parallelogram, find the area of one of the triangles inside the parallelogram.

Area of a parallelogram is 24 sq. feet

First, recall that the area of a parallelogram is base times height and a triangle is one half the area of a parallelogram.

Next, substitute the area value for the base and height in the triangle area formula.

\[ A = \frac{1}{2}bh \]
\[ A = \frac{1}{2}(24) \]

Then, solve the equation.

\[ A = \frac{1}{2}(24) \]
\[ A = 12 \text{ square feet} \]

The answer is 12 square feet.

**Example 5**

Using the area of the following parallelogram, find the area of one of the triangles inside the parallelogram.

Area of a parallelogram is 18 sq. feet

First, substitute the area value for the base and height in the triangle area formula.

\[ A = \frac{1}{2}bh \]
\[ A = \frac{1}{2}(18) \]

Then, solve the equation.

\[ A = \frac{1}{2}(18) \]
\[ A = 9 \text{ square feet} \]

The answer is 9 square feet.
8.5. Triangle Area

Review

Find the area of each triangle given the following measurements.

1. Base = 10 in, Height = 4 in

Base = 16 meters, Height = 10 meters

2. Base = 16 meters, Height = 10 meters
3. Base = 8 in, Height = 6.5 in

4. Base = 10 cm, Height = 7 cm
5. Base = 5 ft, Height = 8.5 feet

Find the area of each triangle given the base and height.

6. Base = 4 in, Height = 5 in  
7. Base = 6 in, Height = 4 in  
8. Base = 8 ft, Height = 7 ft  
9. Base = 10 meters, Height = 8 meters  
10. Base = 10 meters, Height = 5 meters  
11. Base = 12 feet, Height = 14 feet
12. Base = 11 feet, Height = 6 feet  
13. Base = 14 inches, Height = 8 inches  
14. Base = 22 feet, Height = 19 feet  
15. Base = 30 cm, Height = 28 cm  
16. Base = 18 inches, Height = 16 inches  
17. Base = 13 meters, Height = 10 meters  
18. Base = 18 meters, Height = 5.5 meters  
19. Base = 12.5 feet, Height = 2.5 feet  
20. Base = 13.75 inches, Height = 1.5 inches

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.3.

Resources

[Image of a triangle with dimensions and calculations]  
[Media: Click image to the left or use the URL below.  
URL: https://www.ck12.org/flx/render/embeddedobject/182002]
8.6 Area of a Parallelogram

Here you’ll learn how to find the area of a parallelogram given its base and height.

**Area of a Parallelogram**

A parallelogram is a quadrilateral whose opposite sides are parallel.

To find the area of a parallelogram, make it into a rectangle.

From this, we see that the area of a parallelogram is the same as the area of a rectangle. The area of a parallelogram is \( A = bh \). The height of a parallelogram is always perpendicular to the base. This means that the sides are not the height.

What if you were given a parallelogram and the size of its base and height? How could you find the amount of space the parallelogram takes up?

**MEDIA**

Click image to the left or use the URL below.

**URL:** https://www.ck12.org/flx/render/embeddedobject/136630
Examples

Example 1

Find the area of the parallelogram.

Area is $15(6) = 90 \text{ in}^2$.

Example 2

Find the area of the parallelogram with a base of 10 m and a height of 12 m.

Area is $10(12) = 120 \text{ m}^2$.

Example 3

Find the area of the parallelogram.

$A = 15 \cdot 8 = 120 \text{ in}^2$

Example 4

If the area of a parallelogram is 56 $\text{units}^2$ and the base is 4 units, what is the height?

Solve for the height in $A = bh$.

$56 \text{ units} = 4h$

$14 \text{ units} = h$

Example 5

If the height of a parallelogram is 12 m and the area is 60 $\text{m}^2$, how wide is the base?

Solve for the base in $A = bh$.

$60 \text{ units} = 12b$

$5 \text{ units} = b$
8.6. Area of a Parallelogram

Review

1. Find the area of a parallelogram with height of 20 m and base of 18 m.
2. Find the area of a parallelogram with height of 12 m and base of 15 m.
3. Find the area of a parallelogram with height of 40 m and base of 33 m.
4. Find the area of a parallelogram with height of 32 m and base of 21 m.
5. Find the area of a parallelogram with height of 25 m and base of 10 m.

Find the area of the parallelogram.

6.

7.

8.
9. \[ \text{Area of the trapezoid} = \frac{1}{2} (b_1 + b_2) \times h \]

10. \[ \text{Area of the trapezoid} = \frac{1}{2} (6 + 3\sqrt{2}) \times 3\sqrt{2} \]

11. \[ \text{Area of the trapezoid} = \frac{1}{2} (7 + \sqrt{5}) \times 1 \]
12.
13. If the area of a parallelogram is $42 \text{ units}^2$ and the base is 6 units, what is the height?
14. If the area of a parallelogram is $48 \text{ units}^2$ and the height is 6 units, what is the base?
15. If the base of a parallelogram is 9 units and the area is $108 \text{ units}^2$, what is the height?
16. If the height of a parallelogram is 11 units and the area is $27.5 \text{ units}^2$, what is the base?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 10.2.
Jerry was hired to paint the front of a neighbor’s shed. Using the illustration below for dimensions, how many square feet of paint does he need?
In this concept, you will learn to find the area of composite shapes.

**Finding the Area of Composite Shapes**

The area of a rectangle is equal to its length times its width: $A = lw$. The area of a triangle is equal to its base times its height divided by 2: $A = \frac{bh}{2}$. In order to find the composite area of two or more shapes, simply find the area of each shape and add them together. The order in which you calculate the areas does not matter, and the **commutative property** states that it does not matter which order you add them in.

Let’s look at an example.

Find the area of the figure below.

First, if it’s not already done for you, divide the area into shapes you can work with.
Next, find the area of the rectangle.

\[ A = lw \]
\[ A = 9(5) \]
\[ A = 45 \text{ in.}^2 \]

Then, find the area of one of the triangles.

In this figure, you will need to find the base first.

To find the base of one triangle, subtract 9 from 22, then divide by 2.

\[ 22 - 9 = 13 \]
\[ 13 ÷ 2 = 6.5 \]

The base of each triangle is 6.5 inches.

Next, calculate the area of one triangle.

\[ A = \frac{bh}{2} \]
\[ A = \frac{(6.5)(5)}{2} \]
\[ A = 16.25 \text{ in.}^2 \]

The last step is to add the three areas together. Remember, there are two triangles.

\[
\text{Composite Area} = \text{Area of Triangle 1} + \text{Area of Rectangle} + \text{Area of Triangle 2} \\
A = A_{T1} + A_{R} + A_{T2} \\
A = 16.25 + 45 + 16.25 \\
A = 77.5
\]

The answer is 77.5 square inches.

**Examples**

**Example 1**

Earlier, you given a problem about Jerry, who was hired to paint the front of a neighbor’s shed.

He needs to figure out how many square feet of paint he’ll use according to the following diagram.
First, calculate the area of the rectangle.

\[ A_R = l \times w \]
\[ A_R = 6 \times 7.5 \]
\[ A_R = 45 \text{ sq ft} \]

The area of the rectangle is 45 square feet.

Next, calculate the area of the triangle.

\[ A_T = \frac{bh}{2} \]
\[ A_T = \frac{(6)(4)}{2} \]
\[ A_T = 12 \text{ ft}^2 \]

Then, add the areas together.

\[ A = A_R + A_T \]
\[ A = 45 \text{ sq ft} + 12 \text{ sq ft} \]
\[ A = 57 \text{ sq ft} \]

The answer is 57 square feet. Jerry needs a can of paint that will cover 57 sq. feet.

**Example 2**

A figure is made up of three triangles. Each triangle has a base of 6 inches and a height of 4 inches. What is the combined area of all three triangles?

First, use the formula to find the area of one triangle.

\[ A = \frac{bh}{2} \]
\[ A = \frac{(6)(4)}{2} \]
Next, calculate. 
\[ A = 12 \text{ sq in} \]

Then, remember that there are three triangles so multiply your answer times 3.

\[
\begin{align*}
AT &= 12 \text{ sq in} \times 3 \\
AT &= 77.5 \text{ sq in}
\end{align*}
\]

The answer is 77.5 square inches.

**Example 3**

A figure is made up of two triangles and one rectangle. Each triangle has a base of 5 inches and a height of 3 inches. The rectangle has a length of 4 inches and a width of 3 inches. What is the total area of the figure?

First, find the area of the rectangle.

\[
\begin{align*}
AR &= lw \\
AR &= 4(3) \\
AR &= 12 \text{ in}^2
\end{align*}
\]

Next, find the area of one of the triangles.

\[
\begin{align*}
AT &= \frac{bh}{2} \\
AT &= \frac{(5)(3)}{2} \\
AT &= 7.5 \text{ in}^2
\end{align*}
\]

Then, add the three areas together. Remember, there are two triangles.

\[
\begin{align*}
A &= AT_1 + AR + AT_2 \\
A &= 7.5 + 12 + 7.5 \\
A &= 27
\end{align*}
\]

The answer is 27 square inches.

**Example 4**

A figure is made up of one triangle and one square. The square and the triangle have the same base length of 8 feet. The height of the triangle is 7 feet. What is the total area of the figure?

First, find the area of the square.

\[
\begin{align*}
AS &= s^2 \\
AS &= 8^2 \\
AS &= 64 \text{ ft}^2
\end{align*}
\]

Next, find the area of the triangle.

\[
\begin{align*}
AT &= \frac{bh}{2} \\
AT &= \frac{(8)(7)}{2} \\
AT &= 28 \text{ ft}^2
\end{align*}
\]
Then, add the areas together.

\[
A = A_S + A_T
\]

\[
A = 64 \text{ sq ft} + 28 \text{ sq ft} \quad A = 92 \text{ sq ft}
\]

The answer is 92 square feet.

**Example 5**

Find the area of the composite figure below.

First, calculate the area of the rectangle.

\[
A_R = lw
\]

\[
A_R = 17 \times 8.5 \quad A_R = 144.5 \text{ sq in}
\]

The area of the rectangle is 144.5 square inches.

Next, calculate the area of the triangle. In order to do this, you first need to recognize that the height is equal to 17 inches minus 8.5 inches, or 8.5 inches.

Then, use the formula to find the area of the triangle.

\[
A_T = \frac{bh}{2}
\]

\[
A_T = \frac{17(8.5)}{2} \quad A_T = 72.25 \text{ in}^2
\]
Then, add the areas together.

\[ A = A_R + A_T \]
\[ A = 144.5 \text{ sq in} + 72.25 \text{ sq in} \]
\[ A = 216.75 \text{ sq in} \]

The answer is 216.75 square inches.

**Review**

Find the area of each combined figure.

1. A figure is made up of a triangle and a square. The square and the triangle have the same base of 7 inches. The triangle has a height of 5 inches, what is the total area of the figure?
2. A figure is made up of a triangle and a rectangle. The triangle has a height of 8 inches and a base of 9 inches. The rectangle has dimensions of 7 inches \( \times \) 9 inches. What is the area of the figure?
3. A figure is made up of four triangles. Each triangle has a base of 7 inches and a height of 9 inches. What is the total area of the figure?
4. A figure is made up of three triangles. Each triangle has a base of 5 feet and a height of 4.5 feet. What is the total area of the figure?
5. A figure is made up of two triangles and a square. The triangles and the square have the same base length of 5 feet. The triangles have a height of 4 feet. What is the total area of the figure?
6. A figure is made up of two triangles and a square. The triangles and the square have the same base length of 8 feet. The triangles have a height of 7 feet. What is the total area of the figure?
7. A figure is made up of one square and one triangle. The square has a side length of 9 feet. The triangle has a base of 7 feet and a height of 6 feet. What is the total area of the figure?
8. A figure is made up of two triangles. The triangles have the same base of 15 inches. One triangle has a height of 9 inches and one has a height of 11 inches. What is the total area of the figure?
9. A figure is made up of a triangle and a rectangle. The triangle has a height of 8.5 inches and a base of 10 inches. The rectangle has dimensions of 9 inches \( \times \) 10 inches. What is the area of the figure?
10. A figure is made up of a triangle and a rectangle. The triangle has a height of 11 inches and a base of 9.5 inches. The rectangle has dimensions of 12 inches \( \times \) 14 inches. What is the area of the figure?

Solve each problem.

11. Julius drew a triangle that had a base of 15 inches and a height of 11 inches. What is the area of the triangle Julius drew?
12. A triangle has an area of 108 square centimeters. If its height is 9 cm, what is its base?
13. What is the height of a triangle whose base is 36 inches and area is 234 square inches?
14. Tina is painting a triangular sign. The height of the sign is 32 feet. The base is 27 feet. How many square feet will Tracy paint?
15. Tina drew a picture of a triangle with a base of 6 inches and a height of 5.5 inches. What is the area of the triangle?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 9.14.

**Resources**
Find the area of each of these shapes.

\[ A = \frac{1}{2}bh \]
\[ \hat{A} = 10(4) \]
\[ \hat{A} = 40 \text{m}^2 \]

Click the image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/183758
Jerry’s assignment in his carpentry class is to build a wooden storage container that will hold at least 48 cubic units of material. Jerry has already started on the rectangular container, which is 4 units wide and 2 units long. Jerry isn’t sure what height the container should be in order to hold a volume of 48 cubic units. How can Jerry use this information to determine the height of the container?

In this concept, you will learn to identify the volume of prisms using unit cubes.

**Finding Volume of Prisms Using Unit Cubes**

*Volume* is the amount of space inside a solid figure.
These cubes make up a rectangular prism. The cubes represent the volume of the prism. This prism is five cubes by two cubes by one cube. In other words, it is five cubes long, by two cubes high by one cube wide. You can multiply each of these values together to get the volume of the rectangular prism.

\[ 5 \times 2 \times 1 = 10 \text{ cubic units} \]

The volume of the rectangular prism is 10 cubic units or units\(^3\). The units are cubic units because you multiplied the units 3 times when you multiplied the height, length, and width.

Here is another prism.

First, to figure out the volume of this prism, identify the measurements.

Length - 6
Width - 6
Height - 6

Next, substitute these values in the formula for volume then solve.

\[
\text{Volume in cubic units} = \text{Length} \times \text{Width} \times \text{Height}
\]

\[
6 \times 6 \times 6 = 216 \text{ cubic units}
\]

The volume of the cube is 216 cubic units.
**Examples**

**Example 1**

Earlier, you were given a problem about Jerry and the storage container he is building. The container is 4 units wide and 2 units long when he learned about the volume requirement. The container needs to have a volume of 48 cubic units. How can Jerry use this information to determine the height the container should be?

First, to figure the height the container should be, identify the given measurements.

- **Length** - 2 units
- **Width** - 4 units
- **Height** - ?
- **Volume** - 48 cubic units

Next, substitute these values into the formula for volume.

\[
\text{Length} \times \text{Width} \times \text{Height} = \text{Volume in cubic units}
\]

\[
2 \times 4 \times H = 48 \text{ cubic units}
\]

Then, multiply.

\[
8(H) = 48
\]

Then, divide both sides by 8.

\[
H = 6 \text{ units}
\]

The container should have a height of 6 units.

**Example 2**

What is the volume of this figure?
First, to figure out the volume of this prism, identify the measurements.
Length - 4 units
Width - 4 units
Height - 3 units
Next, substitute these values into the formula for volume then solve.

\[
\text{Length} \times \text{Width} \times \text{Height} = \text{Volume in cubic units}\\
4 \times 4 \times 3 = 216 \text{ cubic units}
\]

The volume of the prism is 48 cubic units, or 48 units\(^3\).

**Example 3**

Find the volume of the prism.
First, to figure out the volume of this prism, identify the measurements.

Length - 1
Width - 2
Height - 5

Next, substitute these values into the formula for volume then solve.
The volume of the prism is 10 cubic units.

**Example 4**

Find the volume of the prism.

First, to figure out the volume of this prism, identify the measurements.
Length - 2 inches
Width - 4 inches
Height - 2 inches

Next, substitute these values into the formula for volume then solve.

\[
\text{Volume in cubic units} = \text{Length} \times \text{Width} \times \text{Height}
\]

\[
1 \times 2 \times 5 = 10 \text{ cubic units}
\]

The volume of the cube is 16 cubic inches.

**Example 5**

Find the volume of the prism.
First, to figure out the volume of this prism, identify the measurements.

Length - 2 inches
Width - 4 inches
Height - 1 inch

Next, substitute these values into the formula for volume then solve.

\[
\text{Volume in cubic units} = \text{Length} \times \text{Width} \times \text{Height}
\]

\[
2 \times 4 \times 1 = 8 \text{ cubic units}
\]

The volume of the prism is 8 cubic inches.

**Review**

Find the volume of each prism.
1. Volume of a cube with side length 5 m:

Volume = side × side × side = 5 m × 5 m × 5 m = 125 m³

2. Volume of a rectangular prism with dimensions 2 in × 4 in × 12 in:

Volume = length × width × height = 2 in × 4 in × 12 in = 96 in³
3.
5.

6.
8.8. Volume of Prisms Using Unit Cubes

7.  

8.  

424
Identify each type of prism.
Review (Answers)

To see the Review answers, open this PDF file and look for section 10.15.
In this concept, you will learn how to figure out the volume of rectangular prisms.

Ben’s mom has grown frustrated with Ben’s extensive Lego collection thrown all over his bedroom floor. She gives him a choice of three different boxes in which to store them. Ben wants to choose the box that will hold the most Legos. While the boxes are different in dimensions, Ben cannot figure out which will hold the most. The box measurements are as follows:

Box A: height = 5 inches, length = 18 inches, width = 10 inches
Box B: height = 8 inches, length = 10 inches, width = 12 inches
Box C: height = 6 inches, length = 14 inches, width = 10 inches.

Which box has the greatest volume?

In this concept, you will learn how to figure out the volume of rectangular prisms.

Finding the Volume of a Rectangular Prism

Volume is the measure of how much three-dimensional space an object takes up or holds.

Imagine a fish aquarium. Its length, width, and height determine how much water the tank will hold. If you fill it with water, the amount of water is the volume that the tank will hold. You measure volume in cubic units, because you are multiplying three dimensions: length, width, and height.

One way to find the volume of a prism is to consider how many unit cubes it can contain. A unit cube is simply a cube measuring one inch, one centimeter, one foot, or whatever unit of measurement you are using, on all sides. Here are some unit cubes.
To use unit cubes to calculate volume, simply count the number of unit cubes that fit into the prism. Begin by counting the number of cubes that cover the bottom of the prism, and then count each layer. Let’s see how this works.

How many cubes do you see here? If you count all of the cubes, you will see that there are 24 cubes in this prism. The volume of this prism is 24 units$^3$ or cubic units.

Find the volume of the following figure using unit cubes.

How many cubes are in this figure? If you count all of the cubes, you will get a total of 48 cubes. The volume of this prism is 48 cubic units or units$^3$.

Find the volume of the prism below.

How many cubes are in this figure? If you count the cubes, you will get a total of 48 cubes. The volume of this prism is 48 cubic units or units$^3$.

If you look carefully, you will see that the volume of the rectangular prism is a function of multiplying the length $\times$ the width $\times$ the height. You just discovered the formula for finding the volume of a rectangular prism. Now let’s refine that formula a little further. Here is the formula.

\[ V = Bh \]

The volume is equal to the $B$, base area of the prism, times the height of the prism.

Let’s look at an example.

Find the volume of the prism below.
Simply put the values for the length, width, and height in for the appropriate variables in the formula, then solve for \( V \), volume.

First find the area of the base. This is the rectangular side on the bottom. Remember, to find the area of a rectangle, multiply the length times the width.

\[
B = lw
\]
\[
B = 16 \times 9
\]
\[
B = 144 \text{ cm}^2
\]

The base area is 144 square centimeters. Now multiply this by the height.

\[
V = Bh
\]
\[
V = 144 \times 4
\]
\[
V = 576 \text{ cm}^3
\]

You can use the following formula for volume of a rectangular prism. This combines the two steps that you completed above:

\[
V = lwh
\]
\[
V = (16)(9)(4)
\]
\[
V = 576 \text{ cm}^3
\]

The volume of this rectangular prism is 576 cubic centimeters.

You can work with the same rectangular prism, but fill it with unit cubes.

You can count the unit cubes here to find the volume of the rectangular prism. However, you save time by using the formula for volume.

Let’s look at another example.

Find the volume of a container with a length of 15 ft, width of 12 ft, and height of 11 ft.
First, plug the values of the dimensions into the formula for volume of a rectangular prism and multiply the values for length and width:

\[ V = lwh \]

\[ V = (15)(12)(11) \]

\[ V = (180)(11) \]

Next, multiply the results by the value for the height:

\[ V = (180)(11) \]

\[ V = 1,980 \]

Then, record the answer including the appropriate unit of measurement:

\[ V = 1,980 \text{ ft}^3 \]

The answer is the container has a volume of 1,980 cubic feet.

**Examples**

**Example 1**

Earlier, you were given a problem about Ben, who is searching for the box that will hold the most Legos. Ben needs to figure out which of the following boxes has the greatest volume.

- Box A: height = 5 inches, length = 18 inches, width = 10 inches
- Box B: height = 8 inches, length = 10 inches, width = 12 inches
- Box C: height = 6 inches, length = 14 inches, width = 10 inches

First, plug the values of the dimensions into the formula for volume of a rectangular prism and multiply the values for length and width:

- Box A: \[ V = (5)(18)(10) \]
  \[ V = (90)(10) \]
- Box B: \[ V = (8)(10)(12) \]
  \[ V = (80)(12) \]
- Box C: \[ V = (6)(14)(10) \]
  \[ V = (84)(10) \]
Next, multiply the results by the value for the height:

Box A:  \( V = (90)(10) \)
\[ V = 900 \]

Box B:  \( V = (80)(12) \)
\[ V = 960 \]

Box C:  \( V = (84)(10) \)
\[ V = 840 \]

Then, record the answer including the appropriate unit of measurement:

Box A:  \( V = 900 \text{ in}^2 \)

Box B:  \( V = 960 \text{ in}^2 \)

Box C:  \( V = 840 \text{ in}^2 \)

The answer is Box B has the greatest volume and therefore can hold the most Legos.

**Example 2**

Carla is cleaning out her fish tank, so she filled the bathtub to the rim with water for her fish to swim in while she empties their tank. If the bathtub is 5.5 feet long, 3.3 feet wide, and 2.2 feet deep, how much water can it hold?

First, plug the values of the dimensions into the formula for volume of a rectangular prism and multiply the values for length and width:

\[
V = lwh \\
V = (5.5)(3.3)(2.2)
\]

\[
V = (18.15)(2.2)
\]

Next, multiply the results by the value for the height:

\[
V = (18.15)(2.2) \\
V = 39.93
\]

Then, record the answer including the appropriate unit of measurement:

\[
V = 39.93 \text{ ft}^3
\]

The answer is Carla’s bathtub can hold 39.93 cubic feet of water.
Example 3

Find the volume of a container with a length of 10 inches, width of 8 inches, and height of 6 inches.

First, plug the values of the dimensions into the formula for volume of a rectangular prism and multiply the values for length and width:

\[ V = lwh \]

\[ V = (10)(8)(6) \]

\[ V = (80)(6) \]

Next, multiply the results by the value for the height:

\[ V = (80)(6) \]

\[ V = 480 \]

Then, record the answer including the appropriate unit of measurement:

\[ V = 480 \text{ in}^3 \]

The answer is the container has a volume of 480 cubic inches.

Example 4

Find the volume of a container with a length of 8 meters, width of 7 meters, and height of 3 meters.

First, plug the values of the dimensions into the formula for volume of a rectangular prism and multiply the values for length and width:

\[ V = lwh \]

\[ V = (8)(7)(3) \]

\[ V = (56)(3) \]

Next, multiply the results by the value for the height:
8.9. Volume of Rectangular Prisms

\[ V = (56)(3) \]
\[ V = 168 \]

Then, record the answer including the appropriate unit of measurement:

\[ V = 168 \text{ m}^3 \]

The answer is the container has a volume of 168 cubic meters.

**Review**

Find the volume of each rectangular prism. Remember to label your answer in cubic units.

1. Length = 5 in, width = 3 in, height = 4 in
2. Length = 7 m, width = 6 m, height = 5 m
3. Length = 8 cm, width = 4 cm, height = 9 cm
4. Length = 8 cm, width = 4 cm, height = 12 cm
5. Length = 10 ft, width = 5 ft, height = 6 ft
6. Length = 9 m, width = 8 m, height = 11 m
7. Length = 5.5 in, width = 3 in, height = 5 in
8. Length = 6.6 cm, width = 5 cm, height = 7 cm
9. Length = 7 ft, width = 4 ft, height = 6 ft
10. Length = 15 m, width = 8 m, height = 10 m
11. Length = 10.5 m, width = 11 m, height = 4 m
12. Length = 12 ft, width = 12 ft, height = 8 ft
13. Length = 16 in, width = 8 in, height = 8 in
14. Length = 12 m, width = 12 m, height = 12 m
15. Length = 24 in, width = 6 in, height = 6 in

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 10.10.

**Resources**

Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/166849
In this concept, you will learn how to graph geometric figures given coordinates of vertices and identify graphed figures.

Parker is taking her first drawing class, though she doesn’t have high hopes. When she walks into class the first day, her art teacher pulls up a picture of a cityscape and tells them that they will all be able to draw this by the end of the week. Her teacher gives them each a copy of this picture and tells them that their first assignment is to break the picture into its underlying shapes. Parker is utterly confused and overwhelmed. How can Parker figure out how to break the picture into shapes?

In this concept, you will learn how to graph geometric figures given coordinates of vertices and identify graphed figures.
**Classifying Polygons in the Coordinate Plane**

A **coordinate grid** is a grid in which points are graphed. It usually has two or more intersecting lines which divide a plane into quadrants, and in which ordered pairs, or coordinates, are defined. It usually has four **quadrants**, or sections, to it.

The **origin** is the place where the two lines intersect. Its coordinates are defined as (0,0).

The **x-axis** is the line running from left to right that has the numbers defined on it and is usually labeled with an "x". The x-coordinate of an ordered pair is found with relation to it. All the points located on the x-axis have a y-coordinate of 0.

The **y-axis** is the central line that runs up-down and is labeled with a "y". Y-coordinates are plotted in reference to this axis. Again, all the x-coordinates of points located on the y-axis are 0.

An **ordered pair** is a list of two numbers in parenthesis, separated by a comma like this: (5,-3). It tells where a point is located on the coordinate plane. The first number is the x-coordinate. It tells you where to go on the x-axis. If it is positive, you go to the right. If it is negative, you go to the left. The second number is the y-coordinate. It tells you where to go on the y-axis. If it is positive, you go up. If it is negative, you go down.

The **vertex** of a shape is the place where two sides of the shape come together. In general, when a shape is defined inside of a coordinate plane, it is defined by the vertices, and then the lines are drawn to connect them.

A **polygon** is any shape made up of rectilineal, or straight, lines. The smallest polygon is a **triangle**, which has three sides. A five-sided figure is a **pentagon**. And many polygons with more sides than five are also named.

A **right angle** is an angle that looks like where the axes on the coordinate plane meet.

**Parallel lines** are lines that will go on forever but will never converge.

The category of four-sided polygons includes:

- the **square**, which has four sides of equal length and its angles are all right angles;
- the **rhombus** which, like the square, has four equal sides, but is "tilty";
- the **rectangle** which has two pairs of sides which are equal and all its angles are right angles;
- the **trapezoid** which may have no equal sides, but it has two lines which are parallel; and
- the parallelogram which has two sets of parallel lines which are equal in length to each other (but like the rhombus, it is "tilty").

In order to graph a figure in the coordinate plane, you just graph each of the vertices and then connect them with straight lines so that none of the lines cross. The number of sides you have is the same as the number of vertices. So a triangle, for example, is defined with three vertices.

Here is an example.

Graph a figure with the coordinates $A(-4, 3) \ B(2, 3) \ C(2, -1) \ D(-4, -1)$. When finished, name the figure that has been drawn on the grid.

First, plot each point on the coordinate grid and then connect the lines.

Next, in order to determine what kind of shape it is, first count the number of vertices.

This figure has four, so it is one of the four-sided shapes.

Then, look to see how many of the sides are equal.

In this case, side $AB=DC$ and $AD=BC$. Since there are two sets of equal sides, this is either a rectangle or a parallelogram.

Finally, check the angles.

In this shape, the angles are right angles, so this is a rectangle.

Here is another example.

Graph and name the following figure with these coordinates $D(1, 3) \ E(5, 3) \ F(7, -1) \ G(1, -1)$.
8.10. Polygon Classification in the Coordinate Plane

First, plot the vertices and connect them.

Next, count the number of vertices.

There are four, so this is a four-sided figure, which you can see from its shape.

Then, look at the sides.

None of the sides are the same length, so this shape must be a trapezoid.

Examples

Example 1

Earlier, you were given a problem about Parker and her math-art panic attack.

Her art teacher gives her a picture of buildings in perspective and tells her to break it into its constituent shapes.

First, she draws a y-axis down the center of the back building in the picture, which looks about halfway across the picture. Then, she draws the x-axis as the horizon line.

Next, she squints at the picture and tries to see it not as buildings, but as shapes, instead. On the right side of the picture, she realizes that the buildings make a triangle with one vertex starting at the origin. And on the left of the y-axis, the other side of buildings make another triangle, also with a vertex at the origin. Even the street, she realizes, makes its own triangle, again with one vertex at the origin. And then the sky finishes it with a forth triangle with a vertex at the origin.

Then, she sees that the rest of the picture is just made up of rectangles. There is one below the street triangle, and one on the left buildings. And one where the tiny back building is.

Parker concludes that she might be able to learn to draw after all.

Example 2

Determine the shape given by the following vertices.

\((-4, 6)(4, 6)(0, -6)\)

First, count the number of vertices.

In this case, there are three vertices. If there are three vertices, then the shape has three sides.
Next, determine which shapes are possible.
There is only one closed shape with three sides, and that is a triangle.
The answer is a triangle.

**Example 3**

Determine the shape given by the following vertices.
\((0, 2)(2, 0)(0, -3)\)

First, count the number of vertices.
In this case, there are three vertices. If there are three vertices, then there are three sides.
Next, determine which shapes are possible.
There is only one closed shape with three sides, and that is a triangle.
The answer is a triangle.

**Example 4**

Determine the shape given by the following vertices.
\((-1, -3)(-5, -3)(-7, 1)(-1, 1)\)

First, count the number of vertices.
In this case, there are four vertices. If there are four vertices, then the shape has four sides.
Next, determine which shapes are possible.
The four-sided figures are: square, rhombus, trapezoid, rectangle, and parallelogram.
Then, plot the vertices on a coordinate plane.
Then, determine the properties of the shape.
In this case, the shape has two parallel lines and two non-parallel lines. The only four sides figure in which that is the case is the trapezoid.
The answer is a trapezoid.

**Example 5**

Determine the shape given by the following vertices.
\((3, 3)(0, 3)(0, 0)(3, 0)\)

First, count the number of vertices.
In this case, there are four vertices. If there are four vertices, then the shape has four sides.
Next, determine which shapes are possible.
The four-sided figures are: square, rhombus, trapezoid, rectangle, and parallelogram.
Then, plot the vertices on a coordinate plane.
Then, determine the properties of the shape.
In this case, the shape has two sets of parallel lines and all the sides are the same length. Additionally, the shape
has four right angles. The square is the shape that meets these criterion.
The answer is a square.

Review

Graph each figure using the vertices. Then name the graphed figure.

1. \(A(-2, 2); B(2, 2); C(2, -2); D(-2, -2)\)
2. \(D(-4, 3); E(-1, 1); F(-4, 1)\)
3. 
4. \((-1, 3)(-5, 3)(-1, 0)(-5, 0)\)
5. 
6. \((0, 6)(2, 6)(0, 10)(2, 10)\)
7. 
8. \((-1, 6)(-1, 8)(-9, 6)(-9, 8)\)
9. \((0, -8)(1, -5)(5, -5)(4, -8)\)
10. \((12, 0)(12, 6)(7, 0)\)

For 11 - 15 draw five of your own figures on a coordinate grid. Write out each set of coordinates and work with a partner to identify each figure using only the coordinates.

Review (Answers)

To see the Review answers, open this PDF file and look for section 11.15.
Here you’ll learn different ways of representing three-dimensional objects in two dimensions. In particular, you’ll learn about cross-sections and nets.

**Cross-Sections and Nets**

While our world is three dimensional, we are used to modeling and thinking about three dimensional objects on paper (in two dimensions). There are a few common ways to help think about three dimensions in two dimensions. One way to “view” a three-dimensional figure in a two-dimensional plane (like on a piece of paper or a computer screen) is to use cross-sections. Another way to “view” a three-dimensional figure in a two-dimensional plane is to use a net.

**Cross-Section:** The intersection of a plane with a solid.

The cross-section of the peach plane and the tetrahedron is a *triangle*.

**Net:** An unfolded, flat representation of the sides of a three-dimensional shape.

It is good to be able to visualize cross sections and nets as the three dimensional objects they represent.

What if you were given a three-dimensional figure like a pyramid and you wanted to know what that figure would look like in two dimensions? What would a flat slice or an unfolded flat representation of that solid look like?

**MEDIA**

Click image to the left or use the URL below.

**URL:** https://www.ck12.org/flx/render/embeddedobject/136775
8.11. Cross-Sections and Nets

Examples

Example 1

Describe the cross section formed by the intersection of the plane and the solid.

Circle

Example 2

Determine what shape is formed by the following net.

Square-based pyramid

Example 3

What is the shape formed by the intersection of the plane and the regular octahedron?

Square
Rhombus

Hexagon

**Example 4**

What kind of figure does this net create?

The net creates a rectangular prism.

**Example 5**

Draw a net of the right triangular prism below.
The net will have two triangles and three rectangles. The rectangles are different sizes and the two triangles are the same.

There are several different nets of any polyhedron. For example, this net could have the triangles anywhere along the top or bottom of the three rectangles.

**Review**

Describe the cross section formed by the intersection of the plane and the solid.

Draw the net for the following solids.
Determine what shape is formed by the following nets.

4.

5.

6.

7.

8.

9.

10.
Review (Answers)

To see the Review answers, open this PDF file and look for section 11.2.

Solve real-world and mathematical problems involving area, surface area, and volume.

6.G.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas V = l w h and V = b h to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

6.G.3 Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

6.G.4 Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Review, connect, apply these earlier standards mentioned earlier in the Flexbook:

Apply and extend previous understandings of arithmetic to algebraic expressions.

6.EE.2 Write, read, and evaluate expressions in which letters stand for numbers.

c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas V = s^3 and A = 6 s^2 to find the volume and surface area of a cube with sides of length s = 1/2.

Reason about and solve one-variable equations and inequalities.

6.EE.5 Understand solving an equation or inequality as a process of answering a question: Which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which $p$, $q$, and $x$ are all nonnegative rational numbers.
Develop understanding of statistical variability.

6.SP.1 Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. For example, “How old am I?” is not a statistical question, but “How old are the students in my school?” is a statistical question because one anticipates variability in students’ ages.

6.SP.2 Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.

6.SP.3 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.
9.1 Identification of Misleading Statistics

Here you’ll learn to identify and analyze misleading statistics.

Let’s Think About It

Ashley has asked all the students in 8th grade whether they had eaten pizza in the last week. She organized her data into a table and then created the graph below.
Ashley showed the graph to her friend, Lakki, who told her it was incorrect. What is wrong with Ashley’s graph?

In this concept, you will learn how to analyze and identify misleading statistics that are represented in graphical displays.

**Guidance**

**Graphs** are visual representations of data. Graphs can be used to present, persuade, or even mislead the viewer. The same set of data can be presented on a graph in different ways. Sometimes, the way that a graph is drawn can present only one side of the statistics.

Graphs that are misleading either try to persuade the reader or inaccurately present data. For example, political groups may show graphs that show drastic economic changes, however upon careful inspection it becomes clear that the graph is poorly drawn or purposely misleading.

When reading, analyzing, and interpreting graphs, you should consider:

- the type of graph used.
- how the axes have been drawn, in particular the scale used.
- the characteristics and similarity of the data presented.

Let’s look at an example.

The two graphs below show the average monthly water temperature in Hawaii. Compare the two graphs and determine which one presents the most reliable view of the data.
First, compare and contrast the different graphs. Determine what is the same and what is different. Based on these characteristics, select which one is more reliable.

The answer is: graph 1 presents a more reliable view of the data. Graph 2 shows drastic changes in the temperature, which is not accurate. The change in temperature does not vary greatly from month to month and the graph should illustrate this.

**Guided Practice**

Explain why the data compared on the bar graph below is misleading. Then propose how to correct the misleading information in the graph.
First, analyze the graph. Look at what kind of data is presented and if it’s related, the values along the horizontal and vertical axes, and what kind of message the graph is trying to send.

The answer is the graph is misleading because each state on the graph has a different size coastline. For example, Florida’s vast amount of coastline may contribute to the fact that it has a far higher incidence of shark attacks than any other U.S. state. As well, California, Florida, and Hawaii are all big beach destination states; therefore the incidence of shark attacks will be greater than states such as Alabama. Lastly, the likelihood of sharks being off the stated coastlines must be considered. Sharks may be more likely to be off certain portions of the Florida coastline, than Alabama. To fix the graph, the data presented should be replaced by more comparable data. Specifically, data that compares the incidence of shark attacks among states that are more geographically similar.

Examples

Example 1

The graph below illustrates the findings of a research study that investigated the success rate of small businesses. State two things that are misleading about this graph.
First, analyze the graph to determine what could be misleading about the data presented. Then write two statements describing the misleading characteristics.

The answer is two characteristics that are misleading are:

- the vertical axis is broken off and does not begin at 0. When axes are drawn like this the data can look more spread out than it really is.
- the three studies reported collected data at different intervals. Said another way, study 1 collected data every year for 6 years, but studies 2 and 3 collected data every other year for 6 years. Study 1 shows a gradual decline in the percentage of small businesses open every year whereas studies 2 and 3 show greater declines every other year.

Example 2

The graph below shows the average home prices from 2014 to 2015. What are two misleading characteristics of this graph?
First, analyze the graph to determine what could be misleading about the data presented. Then write two statements describing what is incorrect in the graph.

The answer is two characteristics that are misleading are:

- the vertical axis is purposely drawn to show a great difference among the values $250,000 and $252,000. This difference is highlighted by the title of the graph, which states there was a huge spike in house prices.
- the data reported in the table is taken out of context. Housing prices can vary widely depending on where they are located. The graph does not provide any detailed information about the housing prices reported, such as location or home size.

**Example 3**

The graph below shows the number of jobs lost by quarter. The describe three things that make this graph misleading.

![Job Loss by Quarter](image)
First, analyze the graph to determine what could be misleading about the data presented. Then write three statements describing what is misleading about the data in the graph.

The answer is three characteristics that are misleading are:

- the data presented is by quarter, however the dates listed are not each quarter. For example, between December 2007 and September 2008 there are 3 quarters between, however between September 2008 and March 2009 there are 2 quarters.
- the data is presented in a line, however the data does not increase by the same amount each time. For example, the difference between 7 mil and 9 mil is 2 mil, but the difference between 9 mil and 13.5 mil is 4.5 mil.
- the vertical axis is broken off, it begins at 6 and ends just above 15. The line is meant to look like it is close to 0 and rises dramatically, however the data does not match this representation.

**Follow Up**

Remember Ashley and her graph?

Ashley created the graph below after asking the other 8th graders if they ate pizza this week.
9.1. Identification of Misleading Statistics

Ashley showed the graph to her friend, Lakki, who told her it was incorrect. What is wrong with her graph?

To answer this question, analyze the graph Ashley created.

First, look at what the graph is saying. This graph should tell the reader how many people or what percent of people ate pizza this week. Ashley’s graph actually says both, which is incorrect.

Next, compare the sizes of the columns to the information contained in the legend. This information should match, but in Ashley’s graph it does not. This is incorrect.

The answer is Ashley’s graph has two inaccuracies.

The first inaccuracy is the information the graph is presenting. The graph should either say how many people ate pizza this past week or what percent of people ate pizza this past week. Ashley should change the data presented in her graph to reflect one of these questions. Ashley should also make sure the graph title is accurate and correct - "Have you ate pizza this week?" is grammatically incorrect.

The second inaccuracy is the values represented in the columns and the legend. The legend states that the percent of people that didn’t eat pizza was 16%, however the graph does not accurately show this in the red column. The two columns are not in proportion to each other, meaning that 16% should be closer to 20 than 0. Ashley should check her calculations to ensure the 16% is accurate. To correct this mistake, Ashley should either correct the red column to show the actual number of people that represent 16% or update the legend to reflect the true percent of people.

Video Review
Explore More

Answer each question regarding misleading data.

1. Is this a misleading graph?
2. What is one thing that makes it a misleading graph?
3. What is one thing that you could do to fix this graph?

The data table below depicts the amount of time students at different grade levels spend on homework and studying. Ensure that the second graph shows that time spent on homework in twelfth grade is triple that of sixth grade.

<table>
<thead>
<tr>
<th>Grade</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>6&lt;sup&gt;th&lt;/sup&gt;</td>
<td>1.75</td>
</tr>
<tr>
<td>7&lt;sup&gt;th&lt;/sup&gt;</td>
<td>2</td>
</tr>
<tr>
<td>8&lt;sup&gt;th&lt;/sup&gt;</td>
<td>2.25</td>
</tr>
<tr>
<td>9&lt;sup&gt;th&lt;/sup&gt;</td>
<td>2.5</td>
</tr>
<tr>
<td>10&lt;sup&gt;th&lt;/sup&gt;</td>
<td>2.75</td>
</tr>
<tr>
<td>11&lt;sup&gt;th&lt;/sup&gt;</td>
<td>3</td>
</tr>
<tr>
<td>12&lt;sup&gt;th&lt;/sup&gt;</td>
<td>3.5</td>
</tr>
</tbody>
</table>

4. Use the data below to create two bar graphs. One that shows the data accurately, that time spent on homework in twelfth grade is double that of sixth grade.
5. Ensure that the second graph shows that time spent on homework in twelfth grade is triple that of sixth grade.
6. If the students doubled the time that they spend on homework in the 7th grade, how many hours would they be spending?

7. If the students in the 11th grade spent half as much time on homework, how many more hours of free time would they gain?

8. True or false. All students spend at least one hour on homework.

The data table below depicts the sales tax rate for several U.S. states.

<table>
<thead>
<tr>
<th>State</th>
<th>Sales Tax Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alaska</td>
<td>0</td>
</tr>
<tr>
<td>Alabama</td>
<td>4.0</td>
</tr>
<tr>
<td>Arizona</td>
<td>5.6</td>
</tr>
<tr>
<td>California</td>
<td>6.25</td>
</tr>
<tr>
<td>New Jersey</td>
<td>7.0</td>
</tr>
</tbody>
</table>

9. Use the information on the data table to create two graphs. One graph should depict the data accurately. On this graph, the sales tax rate for New Jersey is almost double the sales tax rate in Alabama.

10. The second graph should present the data in a misleading manner to suggest that the sales tax rate in New Jersey is more than triple the tax rate in Alabama.

11. Which state has the highest state tax?

12. If Alaska doesn’t have a state tax, does it make sense to put it on the list?

The data below depicts the daily temperature in Juneau, Alaska for ten days.

64 60 57 55 49 57 58 60 59 56

13. Draw a line graph that depicts a sharp decrease in temperature.

14. Draw another line graph that depicts the decrease accurately.

15. What is the highest temperature on the list?

16. What is the lowest temperature on the list?

17. - 20. Look through a newspaper and choose three different graphs. Then write a few sentences about each one explaining how the data represented is or is not misleading and why.

**Answers for Explore More Problems**

To view the Explore More answers, open this [PDF file](#) and look for section 11.13.
Here you’ll learn to find the range of a set of data.

Let’s Think About It

Bob the weatherman records the high temperature each day for two weeks in June. He wants to see how much the temperature changes throughout those two weeks. Here is the data he collected for the 14 days he was keeping track of temperature highs:

72, 75, 68, 70, 67, 76, 71, 72, 68, 75, 71, 75, 69, 70

In order for Bob to know the difference in high temperatures throughout the two weeks, he needs to find the range of the data.

In this concept, you will learn how to find the range of a set of data.

Guidance

The range of a set of data simply tells where the numbers fall and whether data in set is close together or spread apart. A set of data with a small range tells us something different than a set of data with a large range.

To find the range of a set of data:
9.2. Range of Spread/Dispersion

1. Put the values in the data set in numerical order. This will show which is the greatest number in the set (the maximum), and which is the smallest number (the minimum).
2. Subtract the minimum from the maximum. This is the range of the data.

Take a look at the data set below.
11, 9, 8, 12, 11, 11, 14, 8, 10
First, arrange the data in numerical order.
8, 8, 9, 10, 11, 11, 11, 12, 14
Next, identify the minimum and maximum of the set. The minimum is 8 and the maximum is 14.
Then, subtract the minimum from the maximum to find the range.

\[14 - 8 = 6\]

The range of the data set is 6. This means that all of the numbers in the data set fall within six places of each other. All of the data results are fairly close together.

The context of a set of data and its range can reveal important information about the data. For example, if a data set showing plant growth using special soil is 4, that shows that the plants all grew in close range to each others. In other words, the special soil had the same impact on all of the plants. Suppose though, the data showed a wider range of 15. That would signify that perhaps the special soil did not have as equal of an impact on the plants as some grew a lot and some grew little to make that range so wide.

**Guided Practice**

The following is the number of patrons at a local movie theater. What is the range of the data?
26, 22, 40, 45, 46, 18, 30, 80, 60, 75
To figure this out, we need to find the difference between the highest number of patrons and the lowest number of patrons.
First, put the data in order from least to greatest.
18, 22, 26, 30, 40, 45, 46, 60, 75, 80
Next, identify the minimum and maximum of the data set. The minimum is 18 and the maximum is 80.
Then, subtract the minimum from the maximum to find the range.

\[80 - 18 = 62\]

The answer is 62. This shows there is a wide range of people who attend the movies in that data set. There is not consistent attendance at the movies.

**Examples**

Find the range of the following data sets.
Example 1

4, 5, 6, 9, 12, 19, 20
First, put the numbers in numerical order. This data set is already in order.
Next, identify the minimum and maximum of the data set. The minimum is 4 and the maximum is 20
Then, subtract the minimum from the maximum to find the range.

\[20 - 4 = 16\]

The answer is 16.

Example 2

5, 2, 1, 6, 8, 20, 25
First, put the numbers in numerical order.
1, 2, 5, 6, 8, 20, 25
Next, identify the minimum and maximum of the data set. The minimum is 1 and the maximum is 25.
Then, subtract the minimum from the maximum to find the range.

\[25 - 1 = 24\]

The answer is 24.

Example 3

65, 23, 22, 45, 11, 88, 99, 123, 125
First, put the numbers in numerical order.
11, 22, 23, 45, 65, 88, 99, 123, 125
Next, identify the minimum and maximum of the data set. The minimum is 11 and the maximum is 125.
Then, subtract the minimum from the maximum to find the range.

\[125 - 11 = 114\]

The answer is 114.
Follow Up

Remember Bob and his weather statistics? Bob kept track of the high temperatures for two weeks to see the difference in temperature. Here was his data:

72, 75, 68, 70, 67, 76, 71, 72, 68, 75, 71, 75, 69, 70

Bob needs to find the range of his data to determine the range in temperature from these two weeks.

First, Bob puts the data in numerical order.

67, 68, 68, 69, 70, 70, 71, 71, 72, 72, 75, 75, 75, 76

Next, Bob identifies the minimum and maximum of the data set. The minimum is 67 and the maximum is 76. These data points mean that both were the high temperatures one day in those two weeks, but the weather clearly changed course at some point.

Then, Bob subtracts the minimum from the maximum.

\[
76 - 67 = 9
\]

The answer is 9. There was a 9 degree change in temperature at some point throughout the two weeks that Bob kept track of high temperatures.

Video Review

Click image to the left or use the URL below.

URL: https://www.ck12.org/flx/render/embeddedobject/54798
Explore More

Find the range for each set of data.

1. 4, 5, 4, 5, 3, 3
2. 6, 7, 8, 3, 2, 4
3. 11, 10, 9, 13, 14, 16
4. 21, 23, 25, 22, 22, 27
5. 27, 29, 29, 32, 30, 32, 31
6. 34, 35, 34, 37, 38, 39, 39
7. 43, 44, 43, 46, 39, 50
8. 122, 100, 134, 156, 144, 110
9. 224, 222, 220, 222, 224, 224
10. 540, 542, 544, 550, 548, 547
11. 2, 3, 3, 2, 2, 2, 5, 6, 7
12. 4, 5, 6, 6, 6, 7, 3, 2
13. 23, 22, 22, 24, 25, 25, 25
14. 123, 120, 121, 120, 121, 125, 121
15. 678, 600, 655, 655, 600, 678, 600, 600

Answers for Explore More Problems

To view the Explore More answers, open this PDF file and look for section 2.21.
9.3 Mean

Here you’ll learn the definition of the mean of a set of numerical data and how to compute the mean of a given set of data as it applies to real-world situations.

Let’s Think About It

Bill coaches a baseball team. He has been keeping track all season of the total runs scored each game that his team plays. This is the list of data he has collected. Each number represents runs scored by his team in each game.

2, 8, 8, 14, 9, 12, 14, 20, 19, 14

Bill wants to find the average amount of runs scored each game over the entire season.

In this concept, you will learn how to find the mean, or average, of a set of number data.

Guidance

One way to analyze a set of number data is to find the mean. Another word for finding the mean is to find the average. An average combines numbers in the data set into one number that best represents the whole set.

There are two steps to finding the mean.

1. Add up all of the numbers in the data set.
2. Divide the total by the number of items (numbers) in the set.

Here is an example. This is a small data set to find the average.

10, 7, 3, 8, 2

First, add all the numbers together.

\[ 10 + 3 + 7 + 8 + 2 = 30 \]

Next, divide the total, 30, by the number of items in the set. There are 5 numbers in the set, so divide 30 by 5.
30 ÷ 5 = 6

The mean, or average, of the set is 6.

**Guided Practice**

Find the average of the data set below.

Jacob has the following quiz scores in his chemistry class for first quarter.
78, 90, 83, 88, 67, 90, 84, 69
What is Jacob’s average for the quarter?

First, add up all of the scores.

\[ 78 + 90 + 83 + 88 + 67 + 90 + 84 + 69 = 649 \]

Next, divide by the number of scores (in other words, the number of quizzes Jacob took).

\[ \frac{649}{8} = 81.1 \]

Then, look at the quotient and decide if the answer should be a whole number rounded answer, or if it can be left as the exact decimal answer. It depends on the context of the data. In this case, all of Jacob’s quiz scores were whole numbers so you can round the answer to a whole number as well.

The answer is that Jacob’s average is an 81.

**Examples**

**Example 1**

Find the mean of the following data:
3, 4, 5, 6, 2, 5, 6, 12, 2

First, add the numbers in the data set together.

\[ 3 + 4 + 5 + 6 + 2 + 5 + 6 + 12 + 2 = 45 \]

Next, divide by the total number of items in the data set.

\[ 45 ÷ 9 = 5 \]

Then, write your answer in either whole number or exact decimal form.

The answer is 5.
Example 2

Find the mean of the following data set:
22, 11, 33, 44, 66, 76, 88, 86, 4
First, add the numbers in the data set together.

\[
22 + 11 + 33 + 44 + 66 + 76 + 88 + 86 + 4 = 430
\]

Next, divide by the total number of items in the data set.

\[
430 \div 9 = 47.77
\]

Then, write your answer in either whole number or exact decimal form.
The answer is 47.8 or round up to 48.

Example 3

Find the mean of the following data set:
37, 123, 234, 567, 321, 909, 909, 900
First, add the numbers in the data set together.

\[
37 + 123 + 234 + 567 + 321 + 909 + 909 + 900 = 4000
\]

Next, divide by the total number of items in the data set.

\[
4000 \div 8 = 500
\]

Then, write your answer in either whole number or exact decimal form.
The answer is 500.
Follow Up

Remember Bill and his baseball team’s scores? He wants to find the average number of runs his team scored throughout the season.
2, 8, 8, 14, 9, 12, 14, 20, 19, 14
First, he adds up the values in the data set.

\[2 + 8 + 8 + 14 + 9 + 12 + 14 + 20 + 19 + 14 = 120\]

Next, Bill divides the total by the number of games his team played this season. They played 10 games.

\[120 \div 10 = 12\]

Then, decide if the answer can be kept actual or needs to be rounded. In this case, the answer comes out as a whole number.
The answer is that Bill’s team scored an average (or mean) of 12 runs each game.

Video Review

Explore More

Find the mean for each set of data. You may round to the nearest tenth when necessary.
1. 4, 5, 4, 5, 3, 3
2. 6, 7, 8, 3, 2, 4
3. 11, 10, 9, 13, 14, 16
4. 21, 23, 25, 22, 22, 27
5. 27, 29, 29, 32, 30, 32, 31
6. 34, 35, 34, 37, 38, 39, 39
7. 43, 44, 43, 46, 39, 50
8. 122, 100, 134, 156, 144, 110
9. 224, 222, 220, 222, 224, 224
10. 540, 542, 544, 550, 548, 547
11. 762, 890, 900, 789, 780, 645, 700
12. 300, 400, 342, 345, 403, 302
13. 200, 199, 203, 255, 245, 230, 211
14. 1009, 1000, 1200, 1209, 1208, 1217
15. 2300, 2456, 2341, 2400, 2541, 2321

**Answers for Explore More Problems**

To view the Explore More answers, open this [PDF file](https://example.com) and look for section 2.18.
Here you’ll learn how to determine the median of a set of numerical data when there is an odd number of values and an even number of values in real-world contexts. You’ll also learn why it is better to use the median than the mean as a measure of central tendency when there are outliers in a set of data.

Let’s Think About It

Sara’s cat has a new litter of kittens each year. She has been keeping track of how many kittens are in each litter. Sara wants to know what the typical amount of kittens is in each litter. To do this, she could find the middle amount of kittens that have been born. Here is Sara’s data. Each number represents an amount of kittens that were born in one year.

7, 8, 9, 12, 14, 10, 15

How can Sara find that middle number in her data set?

In this concept, you will learn how to find the median of a set of data.

Guidance

The median of a set of data is the middle score or number of the data. Median can be useful in examples like finding the median amount of money or median time a runner takes to run a race.

Here is a set of data. To find the median there are a few steps.

2, 5, 6, 2, 8, 11, 13, 14, 15, 21, 22, 25, 27

First, write the numbers in order from the least to greatest. Be sure to include repeated numbers in the list.

2, 2, 5, 6, 8, 11, 13, 14, 15, 21, 22, 25, 27

Next, find the middle number of the set of data.

In this set, there is an odd number of values in the set. There are thirteen numbers in the set. Count 6 on one side of the median and 6 on the other side of the median.

The answer is 13.
This set of data was easy to work with because there was an odd number of values in the set. It does not always work out that way, though. Sometimes, there is an even number of items in the data set.

4, 5, 12, 14, 16, 18

Here, there are six values in the data set. They are already written in order from least to greatest. This data set has two values in the middle because there are six values.

4, 5, 12, 14, 16, 18

The two middle values are 12 and 14. To find the median, add the two middle values together and divide by 2. This is basically taking an average of the two middle values.

\[
\begin{align*}
12 + 14 &= 26 \\
26 \div 2 &= 13
\end{align*}
\]

The median score is 13.

**Guided Practice**

Jess has planted a garden. His big crop has been eggplant. Jess harvested the following numbers of eggplant over five days. What is the median number of eggplant harvested?

12, 9, 15, 6, 9

First, write the numbers in order from least to greatest.

6, 9, 9, 12, 15

Next, count two numbers on each side of the middle of the data set to narrow down to the median number.

Then, find the middle score.

The answer is 9 eggplants in the median number harvested.

**Examples**

Find the median of each data set.

**Example 1**

11, 5, 8, 6, 15

First, order the numbers from least to greatest.

5, 6, 8, 11, 15

Next, count the total number of items in the data set to determine how many to count in from each side.

Then, count in until the median is reached.

The answer is 8.

**Example 2**

4, 1, 6, 9, 2, 11
First, order the numbers from least to greatest.
1, 2, 4, 6, 9, 11

Next, count the total number of items in the data set to determine how many to count in. In this case, there are 6 numbers in the data set, which means an even number of items.

Then, since there are an even number of values in the data set, an average needs to be taken of the two middle values.

\[
\frac{4 + 6}{2} = 5
\]

The answer is 5.

**Example 3**

63, 23, 78, 34, 56, 89

First, order the numbers from least to greatest.
23, 34, 56, 63, 78, 89

Next, count the total number of items in the data set. There is an even number of items in the data set.

Then, since there are 6 values in the set, take an average of the middle two values to determine the median.

\[
\frac{56 + 63}{2} = 59.5
\]

The answer is 59.5

**Follow Up**

Remember Sara and her kittens? She wants to find the middle value of a data set, which is known as finding the median. Here is Sara’s data that shows how many kittens were born in each litter:
7, 8, 9, 12, 14, 10, 15, 14

First, Sara needs to put her data in order from least to greatest.
7, 8, 9, 10, 12, 14, 14, 15
Next, Sara needs to count the number of items in her data set in order to continue on following the correct steps. There are 8 values in her data set, an even number. Then, Sara has to find the two middle values, since there is an even number, and then find the average of those two middle values.

\[
\begin{align*}
10 + 12 &= 22 \\
22 \div 2 &= 11
\end{align*}
\]

The answer is 11. The median number of kittens born each year is 11 per litter.

**Video Review**

Click image to the left or use the URL below.

**MEDIA**

Click image to the left or use the URL below.

**URL:** [https://www.ck12.org/flx/render/embeddedobject/5315](https://www.ck12.org/flx/render/embeddedobject/5315)

**Explore More**

Find the median for each pair of numbers.
1. 16 and 19
2. 4 and 5
3. 22 and 29
4. 27 and 32
5. 18 and 24

Find the median for each set of numbers.
6. 4, 5, 4, 5, 3, 3
7. 6, 7, 8, 3, 2, 4
8. 11, 10, 9, 13, 14, 16
9. 21, 23, 25, 22, 22, 27
10. 27, 29, 29, 32, 30, 32, 31
11. 34, 35, 34, 37, 38, 39, 39
12. 43, 44, 43, 46, 39, 50
13. 122, 100, 134, 156, 144, 110
14. 224, 222, 220, 222, 224, 224
15. 540, 542, 544, 550, 548, 547
Answers for Explore More Problems

To view the Explore More answers, open this PDF file and look for section 2.19.
You’ll learn how to identify the mode or modes of a data set for both quantitative and qualitative data given different representations. You’ll also learn to classify distribution of data as being unimodal, bimodal, or multimodal.

Let’s Think About It

Clark plays on his high school basketball team. Every game, their player statistics are kept track of and Clark reviews the amount of points he scores each game. Here are Clark’s points scored for each game this season:

12, 17, 11, 13, 14, 22, 11, 13, 17, 13, 14, 13, 19, 12

Clark wants to know which amount of points he scored most often this season. To do this, Clark needs to see which number of points shows up most often in his data set.

In this concept, you will learn how to find the mode of a set of data.

Guidance

The mode of a set of data is simply the number that occurs most often. Putting the data in numerical order makes it easy to see how often each value in the data set occurs.

Here is an example:

61, 54, 60, 59, 54, 51, 60, 53, 54

First, put the data in numerical order.

51, 53, 54, 54, 54, 59, 60, 60, 61

Next, look for any numbers that repeat. Both 54 and 60 appear in the data set more than once. The mode is only which value appears the most.

Then, choose the value that appears most in the data set.

The answer is 54.

If there aren’t any numbers that occur more than once, or if numbers appear in the set the same number of times, the set has no mode.

22, 19, 19, 16, 18, 21, 30, 16, 27
In the set above, both 16 and 19 occur twice. No number in the set happens the most often, so there is no mode for this set.

**Guided Practice**

Suppose the data below shows how many people visit the zoo each afternoon. What amount of people is the most frequent number of visitors. In other words, what is the mode of the data set?

68, 104, 91, 80, 91, 65, 90, 91, 70, 80

First, put the numbers in numerical order.

65, 68, 70, 80, 80, 90, 91, 91, 91, 104

Next, look for numbers in the data set that repeat. Both 80 and 91 repeat in the data set.

Then, choose the number in the data set that occurs the most. 91 occurs 3 times whereas 80 occurs 2 times.

The answer is 91.

**Examples**

Find the mode of each data set.

**Example 1**

2, 4, 4, 4, 6, 7, 8, 8, 10, 10, 11, 12

First, put the data in numerical order. This data set is already in numerical order.

Next, look for repeating numbers in the data set. Multiple numbers repeat in this data set: 4, 8, 10.

Then, choose the number that occurs most in the data set. Four occurs more than 8 and 10 does.

The answer is 4.

**Example 2**

5, 8, 9, 1, 2, 9, 8, 10, 11, 18, 19, 20

First, put the data in numerical order.

1, 2, 5, 8, 8, 9, 9, 10, 11, 18, 19, 20

Next, look for repeating numbers in the data set. The numbers 8 and 9 both repeat in the set.

Then, choose the number that occurs most in the data set. In this case, 8 and 9 occur the same number of times; therefore, there is no mode for a number that occurs most often.

The answer is there is no mode.

**Example 3**

12, 12, 5, 6, 7, 11, 23, 23, 67, 23, 89, 23

First, put the data in numerical order.

5, 6, 7, 11, 12, 12, 23, 23, 23, 23, 67, 89
Next, look for repeating numbers in the data set. The numbers 12 and 23 both repeat in this data set. Then, choose the number that occurs most in the data set. The number 12 occurs twice while 23 repeats four times. The answer is 23.

**Follow Up**

Remember Clark and his basketball points? He wants to figure out which number of points in a game he has scored most often. Here are Clark’s statistics:

12, 17, 11, 13, 14, 22, 11, 13, 17, 13, 14, 13, 19, 12

What is the mode of Clark’s data?

First, Clark puts his points in numerical order from least to greatest.

11, 11, 12, 12, 13, 13, 13, 13, 14, 14, 17, 17, 19, 22

Next, Clark looks for repeating scores in his data set. Clark has multiple points that he scored in more than one game: 11, 12, 13, 14, and 17.

Then, Clark looks for the scored points that occurs most often in his data set. Clark scored 13 points in 4 different games.

The answer is 13. This is the point value Clark scored in games most often.
Explore More

Identify the mode for the following sets of data.
1. 2, 3, 3, 3, 2, 2, 2, 5, 6, 7
2. 4, 5, 6, 6, 6, 7, 3, 2
3. 23, 22, 22, 24, 25, 25, 25
4. 123, 120, 121, 120, 121, 125, 121
5. 678, 600, 655, 655, 600, 678, 600, 600
6. 4, 5, 4, 5, 3, 3
7. 6, 7, 8, 3, 2, 4, 4, 7, 7, 7
8. 11, 10, 9, 13, 14, 16, 11, 10, 11
9. 21, 23, 25, 22, 22, 22, 27
10. 27, 29, 29, 32, 30, 32, 31
11. 34, 35, 34, 37, 38, 39, 39, 34, 34
12. 43, 44, 43, 46, 39, 50, 43, 43
13. 122, 100, 134, 156, 144, 110, 110
14. 224, 222, 220, 222, 224, 224
15. 540, 542, 544, 550, 548, 547, 547, 550, 550

Answers for Explore More Problems

To view the Explore More answers, open this PDF file and look for section 2.20.

Develop understanding of statistical variability.

6.SP.1 Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. For example, “How old am I?” is not a statistical question, but “How old are the students in my school?” is a statistical question because one anticipates variability in students’ ages.

6.SP.2 Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.

6.SP.3 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.
Chapter 10: Module 6: Statistics (Topic B)

Chapter Outline

10.1 QUARTILES
10.2 LINE GRAPHS
10.3 BROKEN-LINE GRAPHS
10.4 LINE PLOTS FROM FREQUENCY TABLES
10.5 LINE GRAPHS TO DISPLAY DATA OVER TIME
10.6 BOX-AND-WHISKER PLOTS
10.7 APPLICATIONS OF BOX-AND-WHISKER PLOTS
10.8 REPRESENT REAL-WORLD DATA USING BAR GRAPHS, FREQUENCY TABLES AND HISTOGRAMS
10.9 FREQUENCY TABLES AND HISTOGRAMS
10.10 HISTOGRAMS
10.11 APPLICATIONS OF HISTOGRAMS
10.12 TRENDS IN DATA
10.13 SURVEYS AND SAMPLES
10.14 LEVELS OF MEASUREMENT
10.15 UNGROUPED DATA TO FIND THE MEAN
10.16 GROUPED DATA TO FIND THE MEAN
10.17 MEDIAN OF LARGE SETS OF DATA

Summarize and describe distributions.

6.SP.4 Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

6.SP.5 Summarize numerical data sets in relation to their context, such as by:

a. Reporting the number of observations.

b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.

c. Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.
10.1 Quartiles

In this concept, you will learn how to find measures of data to build a box-and-whisker plot.

William is making a sales report for the week. The number of items sold were as follows:
Monday - 42
Tuesday - 32
Wednesday - 80
Thursday - 75
Friday - 90
Saturday - 65
Sunday - 22
What measure of data does William need to build a box-and-whisker plot for his report?

In this concept, you will learn how to find measures of data to build a box-and-whisker plot.

Quartiles

Once you have collected data, you can graph the information with a box-and-whisker plot. Before building a box-and-whisker plot, you will need to find the median, quartile, and extremes of the data.

The **median** is the middle number in a set of data that is ordered from least to greatest. If there is an odd number of values, the middle number is the median. If there is an even number of values, the average of the two middle values is the median.

Here is a data set from a survey of the number of hours worked by teenagers with part-time jobs.
16, 10, 8, 8, 11, 11, 12, 15, 10, 20, 6, 16, 8

First, order the data from least to greatest including any repeated numbers.
6, 8, 8, 8, 10, 10, 11, 11, 12, 15, 16, 16, 20

Then, identify the median. There are 13 values in the data set. Remember that the median is the middle number in a data set with an odd number of values. The median is 11.
6, 8, 8, 8, 10, 10, 11, **11**, 12, 15, 16, 16, 20

Use the median to divide the data set into two parts. The first half are the values from 6 to 10 and the second half are the values starting at the second 11 up to 20.
6, 8, 8, 8, 10, 10, **11**, 11, 12, 15, 16, 16, 20

A **quartile** divides the data set into 4 parts. The median of the first half of the data is called the **lower quartile**. The median of the second half of the data is called the **upper quartile**.

The lower quartile is the average between 8 and 8. The lower quartile is 8.
The upper quartile is the average between 15 and 16. The upper quartile is 15.5.
10.1. Quartiles

6, 8, 8 | 10, 10, 11, 11, 12, 15 | 16, 16, 20

The **extremes** are the lowest value in a data set (the **lower extreme**) and the highest value in a data set (the **upper extreme**).

In the data set, 6 is the lower extreme and 20 is the upper extreme.

This information will be used to make a box-and-whisker plot.

**Examples**

**Example 1**

Earlier, you were given a problem about William’s sales report.

William needs to identify the measure of data to build a box-and-whisker plot for his sales report.

Monday - 42
Tuesday - 32
Wednesday - 80
Thursday - 75
Friday - 90
Saturday - 65
Sunday - 22

Find the median, lower quartile, upper quartile, lower extreme, and upper extreme of the data set.

First, order the data values from least to greatest.

22, 32, 42, 65, 75, 80, 90

Then, identify the median. The middle value is 65.

Next, identify the lower and upper quartile. The median of the first data set is 32. The median of the second data set is 80.

After that, identify the lower and upper extremes. The lowest value is 22 and the highest value is 90.

The measures of data are as follows.

median: 65
lower quartile: 32
upper quartile: 80
lower extreme: 22
upper extreme: 90

**Example 2**

What is the median of this data set?

4, 5, 12, 11, 9, 8, 7, 4, 3

First, order the values from least to greatest.

3, 4, 4, 5, 7, 8, 9, 11, 12
Then, identify the median. There are nine values in this data set. The median is the middle number.

3, 4, 4, 5, 7, 8, 9, 11, 12

The median is 7.

**Example 3**

What is the median of this data set?

4, 4, 5, 6, 7, 8, 11, 13, 16

The values are already listed in order from least to greatest.

Identify the middle value.

4, 4, 5, 6, 7, 8, 11, 13, 16

The median is 7.

**Example 4**

What is the lower and upper quartile of this data set?

4, 4, 5, 6, 7, 8, 11, 13, 16

First, identify the median of the first half of the data set.

4, 4 | 5, 6, 7, 8, 11, 13, 16

The lower quartile is between 4 and 5.

Then, identify the median of the second half of the data set.

4, 4 | 5, 6, 7, 8, 11 | 13, 16

The upper quartile is between 11 and 13.

The lower quartile is 4.5 and the upper quartile is 12.

**Example 5**

What are the extremes of this data set?

4, 4, 5, 6, 7, 8, 11, 13, 16

The lowest value is 4 and the highest value is 16.

The lower extreme is 4 and the upper extreme is 16.

**Review**

Use each data set to answer the questions following it.

3, 5, 6, 8, 11, 13, 15, 17, 19

1. How many values are there in this data set?
2. What is the median of the data?
3. What is the range?
4. What is the upper quartile?
5. What is the lower quartile?
6. What are the extremes?

100, 112, 115, 122, 123, 126, 130, 131

7. How many values are there in this data set?
8. What is the median of the data?
9. What is the range?
10. What is the upper quartile?
11. What is the lower quartile?
12. What are the extremes?

113, 120, 131, 142, 150, 155, 157, 161, 167

13. How many values are there in this data set?
14. What is the median of the data?
15. What is the range?
16. What is the upper quartile?
17. What is the lower quartile?
18. What are the extremes?

**Review (Answers)**

To see the Review answers, open this [PDF file](https://www.ck12.org/flx/render/embeddedobject/40) and look for section 6.16.

**Resources**

- [Click image](https://www.ck12.org/flx/render/embeddedobject/40) to the left or use the URL below.
- URL: [https://www.ck12.org/flx/render/embeddedobject/40](https://www.ck12.org/flx/render/embeddedobject/40)
Here you’ll learn the difference between continuous data and discrete data as it applies to a line graph. You’ll also learn how to represent data that has a linear pattern on a graph and how to solve problems with line graphs.

**Line Graphs**

Before you continue to explore the concept of representing data graphically, it is very important to understand the meaning of some basic terms that will often be used in this concept. The first such definition is that of a variable. In statistics, a variable is simply a characteristic that is being studied. This characteristic assumes different values for different elements, or members, of the population, whether it is the entire population or a sample. The value of the variable is referred to as an observation, or a measurement. A collection of these observations of the variable is a data set.

Variables can be quantitative or qualitative. A *quantitative variable* is one that can be measured numerically. Some examples of a quantitative variable are wages, prices, weights, numbers of vehicles, and numbers of goals. All of these examples can be expressed numerically. A quantitative variable can be classified as discrete or continuous. A *discrete variable* is one whose values are all countable and does not include any values between 2 consecutive values of a data set. An example of a discrete variable is the number of goals scored by a team during a hockey game. A *continuous variable* is one that can assume any countable value, as well as all the values between 2 consecutive numbers of a data set. An example of a continuous variable is the number of gallons of gasoline used during a trip to the beach.

A *qualitative variable* is one that cannot be measured numerically but can be placed in a category. Some examples of a qualitative variable are months of the year, hair color, color of cars, a person’s status, and favorite vacation spots. The following flow chart should help you to better understand the above terms.

Variables can also be classified as dependent or independent. When there is a linear relationship between 2 variables, the values of one variable depend upon the values of the other variable. In a linear relation, the values of $y$ depend upon the values of $x$. Therefore, the **dependent variable** is represented by the values that are plotted on the $y$-axis, and the **independent variable** is represented by the values that are plotted on the $x$-axis.
Linear graphs are important in statistics when several data sets are used to represent information about a single topic. An example would be data sets that represent different plans available for cell phone users. These data sets can be plotted on the same grid. The resulting graph will show intersection points for the plans. These intersection points indicate a coordinate where 2 plans are equal. An observer can easily interpret the graph to decide which plan is best, and when. If the observer is trying to choose a plan to use, the choice can be made easier by seeing a graphical representation of the data.

**Describing Variables**

Select the best descriptions for the following variables and indicate your selections by marking an ‘x’ in the appropriate boxes.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Quantitative</th>
<th>Qualitative</th>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of members in a family</td>
<td></td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>A person’s marital status</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of a person’s arm</td>
<td></td>
<td>x</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Color of cars</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of errors on a math test</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The variables can be described as follows:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Quantitative</th>
<th>Qualitative</th>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of members in a family</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A person’s marital status</td>
<td></td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of a person’s arm</td>
<td></td>
<td>x</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Color of cars</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of errors on a math test</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>
Creating a Table of Values

Sally works at the local ballpark stadium selling lemonade. She is paid $15.00 each time she works, plus $0.75 for each glass of lemonade she sells. Create a table of values to represent Sally’s earnings if she sells 8 glasses of lemonade. Use this table of values to represent her earnings on a graph.

The first step is to write an equation to represent her earnings and then to use this equation to create a table of values. $y = 0.75x + 15$, where $y$ represents her earnings and $x$ represents the number of glasses of lemonade she sells.

<table>
<thead>
<tr>
<th>Number of Glasses of Lemonade</th>
<th>Earnings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$15.00</td>
</tr>
<tr>
<td>1</td>
<td>$15.75</td>
</tr>
<tr>
<td>2</td>
<td>$16.50</td>
</tr>
<tr>
<td>3</td>
<td>$17.25</td>
</tr>
<tr>
<td>4</td>
<td>$18.00</td>
</tr>
<tr>
<td>5</td>
<td>$18.75</td>
</tr>
<tr>
<td>6</td>
<td>$19.50</td>
</tr>
<tr>
<td>7</td>
<td>$20.25</td>
</tr>
<tr>
<td>8</td>
<td>$21.00</td>
</tr>
</tbody>
</table>

The dependent variable is the money earned, and the independent variable is the number of glasses of lemonade sold. Therefore, money is on the $y$-axis, and the number of glasses of lemonade is on the $x$-axis.

From the table of values, Sally will earn $21.00 if she sells 8 glasses of lemonade.

Now that the points have been plotted, the decision has to be made as to whether or not to join them. Between every 2 points plotted on the graph are an infinite number of values. If these values are meaningful to the problem, then the plotted points can be joined. This type of data is called **continuous data**. If the values between the 2 plotted points are not meaningful to the problem, then the points should not be joined. This type of data is called **discrete data**. Since glasses of lemonade are represented by whole numbers, and since fractions or decimals are not appropriate values, the points between 2 consecutive values are not meaningful in this problem. Therefore, the points should not be joined. The data is discrete.
Interpreting a Graph

The following graph represents 3 plans that are available to customers interested in hiring a maintenance company to tend to their lawn. Using the graph, explain when it would be best to use each plan for lawn maintenance.

From the graph, the base fee that is charged for each plan is obvious. These values are found on the y-axis. Plan A charges a base fee of $200.00, Plan C charges a base fee of $100.00, and Plan B charges a base fee of $50.00. The cost per hour can be calculated by using the values of the intersection points and the base fee in the equation $y = mx + b$ and solving for $m$. Plan B is the best plan to choose if the lawn maintenance takes less than 12.5 hours. At 12.5 hours, Plan B and Plan C both cost $150.00 for lawn maintenance. After 12.5 hours, Plan C is the best deal, until 50 hours of lawn maintenance is needed. At 50 hours, Plan A and Plan C both cost $300.00 for lawn maintenance. For more than 50 hours of lawn maintenance, Plan A is the best plan. All of the above information was obvious from the graph and would enhance the decision-making process for any interested client.
participant skates, up to a maximum of 6 hours. Create a table of values and draw a graph to represent a participant who skates for the entire 6 hours. How much money can a participant raise for the community if he/she skates for the maximum length of time?

The equation for this scenario is \( y = 3x + 10 \), where \( y \) represents the money made by the participant, and \( x \) represents the number of hours the participant skates.

<table>
<thead>
<tr>
<th>Numbers of Hours Skating</th>
<th>Money Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10.00</td>
</tr>
<tr>
<td>1</td>
<td>$13.00</td>
</tr>
<tr>
<td>2</td>
<td>$16.00</td>
</tr>
<tr>
<td>3</td>
<td>$19.00</td>
</tr>
<tr>
<td>4</td>
<td>$22.00</td>
</tr>
<tr>
<td>5</td>
<td>$25.00</td>
</tr>
<tr>
<td>6</td>
<td>$28.00</td>
</tr>
</tbody>
</table>

The dependent variable is the money made, and the independent variable is the number of hours the participant skated. Therefore, money is on the \( y \)-axis, and time is on the \( x \)-axis as shown below:

A participant who skates for the entire 6 hours can make $28.00 for the "Skate for Scoliosis" event. The points are joined, because the fractions and decimals between 2 consecutive points are meaningful for this problem. A participant could skate for 30 minutes, and the arena would pay that skater $1.50 for the time skating. The data is continuous.

**Review**

1. What term is used to describe a data set in which all points between 2 consecutive points are meaningful?
   a. discrete data
   b. continuous data
   c. random data
   d. fractional data
2. What type of variable is represented by the number of pets owned by families?
   a. qualitative
   b. quantitative
   c. independent
   d. continuous

3. What type of data, when plotted on a graph, does not have the points joined?
   a. discrete data
   b. continuous data
   c. random data
   d. independent data

4. Select the best descriptions for the following variables and indicate your selections by marking an 'x' in the appropriate boxes.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Quantitative</th>
<th>Qualitative</th>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men’s favorite TV shows</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Salaries of baseball players</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of children in a family</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Favorite color of cars</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of hours worked weekly</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

You are selling your motorcycle, and you decide to advertise it on the Internet on Walton’s Web Ads. He has 3 plans from which you may choose. The plans are shown on the following graph. Use the graph and explain when it is best to use each plan.

5. When would it be best to use Plan A?
6. When would it be best to use Plan B?
7. When would it be best to use Plan C?
8. What is the dependent variable in the following relationship? The time it takes to run the 100 yard dash and the fitness level of the runner.
   a. fitness level
   b. time
   c. length of the track
   d. age of the runner
9. If the relationship in question 8 were graphed on a coordinate grid, what variable would be on the x-axis?
10. If the relationship in question 8 were graphed on a coordinate grid, what variable would be on the y-axis?

**Review (Answers)**

To view the Review answers, open this PDF file and look for section 7.1.
Here you’ll learn what type of data can be represented on broken-line graphs, how to construct broken-line graphs, and how to interpret other data from the finished graphs.

**Broken Line Graphs**

A variation of a line graph is a **broken-line graph**. This type of line graph is used when it is necessary to show change over time. A line is used to join the values, but the line has no defined slope. However, the points are meaningful, and they all represent an important part of the graph. Usually a broken-line graph is given to you, and you must interpret the given information from the graph.

**Interpreting a Broken Line Graph**

1. Answer the questions below for the following broken-line graph, which shows the distance, over time, of a bus from the bus depot.

a. What was the fastest speed of the bus?
The fastest speed of the bus was 16 miles per hour.

b. How many times did the bus stop on its trip? (Do not count the beginning and the end of the trip.)
The bus was stopped 4 times.

c. What was the initial distance of the bus from the bus depot?
The bus was initially 2 miles from the bus depot.

d. What was the total distance traveled by the bus?
The total distance traveled by the bus was 38 miles.

2. Sam decides to spend some time with his friend Aaron. He hops on his bike and starts off to Aaron’s house, but on his way, he gets a flat tire and must walk the remaining distance. Once he arrives at Aaron’s house, they repair the flat tire, play some poker, and then Sam returns home. On his way home, Sam decides to stop at the mall to buy a book on how to play poker. The following graph represents Sam’s adventure:

![Graph showing Sam's adventure]

a) How far is it from Sam’s house to Aaron’s house?
It is 25 km from Sam’s house to Aaron’s house.

b) How far is it from Aaron’s house to the mall?
It is 15 km from Aaron’s house to the mall.

c) At what time did Sam have a flat tire?
Sam had a flat tire at 10:00 am.

d) How long did Sam stay at Aaron’s house?
Sam stayed at Aaron’s house for 1 hour.

e) At what speed did Sam travel from Aaron’s house to the mall and then from the mall to home?
Sam traveled at a speed of 30 km/h from Aaron’s house to the mall and then at a speed of 40 km/h from the mall to home.

3. The following graph is an example of a broken-line graph, and it represents the time of a round-trip journey, driving from home to a popular campground and back.
a) How far is it from home to the picnic park?
   It is 40 miles from home to the picnic park.

b) How far is it from the picnic park to the campground?
   It is 60 miles from the picnic park to the campground.

c) At what 2 places did the car stop?
   The car stopped at the picnic park and at the campground.

d) How long was the car stopped at the campground?
   The car was stopped at the campground for 15 minutes.

f) When does the car arrive at the picnic park?
   The car arrived at the picnic park at 11:00 am.

g) How long did it take for the return trip?
   The return trip took 1 hour.

h) What was the speed of the car from home to the picnic park?
   The speed of the car from home to the picnic park was 40 mi/h

h) What was the speed of the car from the campground to home?
   The speed of the car from the campground to home was 100 mi/h.
Example

Example 1

For the following broken-line graph, write a story to accompany the graph, and provide a detailed description of the events that are occurring.

Before we write our story, let’s summarize what we know from the graph. We know that the story begins at 8:45 am, because the line touches the x-axis at the third tick mark between 8:00 am and 9:00 am. We also know that from 8:45 am to 9:30 am, a distance of 60 km was traveled. This is because from 8:45 am to the second tick mark between 9:00 and 10:00 am, the line goes from 0 km to 60 km on the y-axis. This means that 60 km was traveled in 45 minutes, so the speed during this time can be calculated as follows:

\[
\frac{60}{45} = \frac{x}{60}
\]

\[
45x = 3,600
\]

\[
x = 80 \text{ km/hr}
\]

Next, we can see that from 9:30 am to 10:00 am, no distance was traveled, since the line is horizontal in this interval. However, from 10:00 am to 11:00 am, 40 km was traveled. This is because from 10:00 am to 11:00 am, the line goes from 60 km to 100 km on the y-axis, and \(100 - 60 = 40\). Therefore, the speed during this time was 40 km/hr. From 11:00 am to 11:15 am, there was, again, no distance traveled, since the line is horizontal from 11:00 am to the first tick mark between 11:00 am and 12:00 pm. Finally, from 11:15 am to 12:15 pm, 100 km was traveled, since the line goes from 100 km to 0 km on the y-axis from 11:15 am to the first tick mark after 12:00 pm. This means that the speed during this time was 100 km/hr. Also, since the distance is decreasing in this interval, we know that this was a return trip. Because the line touches the x-axis at 12:15 pm, this is the end of the trip.

Now we are ready to write our story. As long as it adheres to what we found above, anything is fine, but here is an example of what we could write:

Deena is going on a shopping trip. From 8:45 am to 9:30 am, she drove her car to a neighboring town 60 km away, traveling at a speed of 80 km/hr. She then decided to stop and have breakfast for 30 minutes before resuming her trip. Traffic was a little heavy for the next hour, so she only managed to go 40 km in this time, traveling at a speed of 40 km/hr. At 11:00 am, she reached the shopping mall that was her destination, but it seemed to be closed. After looking around for 15 minutes, she decided that it was, in fact, closed, so she began her trip home. Traffic was much
lighter on the way home, so she covered the entire 100 km non-stop in 1 hour, traveling at a speed of 100 kn/hr. She arrived home at 12:15 pm.

**Review**

1. What name is given to a graph that shows change over time, with points that are joined but have no defined slope?

   a. linear graph  
   b. broken-line graph  
   c. scatter plot  
   d. line of best fit

Use the broken-line graph below, which represents a bike ride, to answer the following questions.

2. What was the total distance traveled on the bike ride?  
3. What was the fastest speed traveled by the bike?  
4. What was the slowest speed traveled by the bike?  
5. How long did the bicyclist stop before beginning his or her return trip?  
6. How long did the return trip take?

Bob is looking for the post office, but he is lost. The broken-line graph below shows his distance from the post office as he wanders about the city. Use the broken-line graph to answer the following questions.
7. During what time intervals is Bob getting closer to the post office?
8. During what time intervals is Bob getting farther away from the post office?
9. What is the total distance traveled by Bob from 12:30 pm to 6:00 pm, which is the duration of time shown by the graph?
10. What was Bob’s average speed from 12:30 pm to 6:00 pm?

**Review (Answers)**

To view the Review answers, open this PDF file and look for section 7.2.
In this concept, you will learn how to take data in a frequency table and create a line plot to display the data.

Mr. Smith took a survey in his class of how many books his students read over the summer. He first created a list of the data as shown here:

1, 1, 2, 2, 2, 3, 4, 4, 5, 6, 6, 6, 7

Each number represents a student and the number of books they read over the summer. He decides to organize the data further into a frequency table.

<table>
<thead>
<tr>
<th># of Books Read</th>
<th>Frequency (# of students who read that many books)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

Now, he wants to display the data visually for his students to analyze. How can Mr. Smith visually organize his frequency table for his students?

In this concept, you will learn how to take data in a frequency table and create a line plot to display the data.

**Line Plots**

A **line plot** is another display method to organize data.

Like a frequency table, a line plot shows how many times each number appears in the data set. Instead of putting the information into a table, it is placed on a number line. Line plots are especially useful when the data falls over a large range. Take a look at the data and the line plot below.
This data represents the number of students in each class at a local community college.
30, 31, 31, 33, 33, 33, 33, 37, 37, 38, 40, 40, 41, 41, 41

The first thing to do is to organize this data into a frequency table. That will show how often each number appears.

**Table 10.7:**

<table>
<thead>
<tr>
<th># of students</th>
<th>Frequency (number of classes with that many students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>31</td>
<td>3</td>
</tr>
<tr>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td>33</td>
<td>4</td>
</tr>
<tr>
<td>34</td>
<td>0</td>
</tr>
<tr>
<td>35</td>
<td>0</td>
</tr>
<tr>
<td>36</td>
<td>0</td>
</tr>
<tr>
<td>37</td>
<td>2</td>
</tr>
<tr>
<td>38</td>
<td>1</td>
</tr>
<tr>
<td>39</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>2</td>
</tr>
<tr>
<td>41</td>
<td>3</td>
</tr>
</tbody>
</table>

Some conclusions can be drawn from this data:

1. The range of students in each class is from 30 to 41.
2. There aren’t any classes with 32, 34, 35, 36 or 39 students in them.

A line plot can be created to display this same data, only in a different way.

Building the line plot involves counting the number of students and then plotting the information on a number line. The symbol $X$ is used to represent the number of classes that has that number of students in it.

```
  x
  x   x
  x   x   x
  x   x   x   x

  30 31 32 33 34 35 36 37 38 39 40 41
```

Notice that even though there is no class with 32 students in it, is was still included on the number line. This is very important. Each value in the range of numbers needs to be represented, even if that value is 0.

**Examples**

**Example 1**

Earlier, you were given a problem about Mr. Smith and his data showing the number of books his students read over the summer. This was the data he collected and then organized into a frequency table:
1, 1, 2, 2, 2, 3, 4, 4, 5, 6, 6, 6, 7
Mr. Smith has done the first step to creating a visual display of data which is to organize the data and create the frequency table.

Next, he creates the line plot.

Then, he can analyze the data and draw conclusions. For example:

- 2 students read only 1 book over the summer
- 1 student read 7 books over the summer
- There were no students who did not read at all- that’s great!

**Example 2**

Jeff counted the number of ducks he saw swimming in the pond each morning on his way to school. Here are his results:

6, 8, 12, 14, 5, 6, 7, 8, 12, 11, 12, 5, 6, 6, 8, 11, 8, 7, 6, 13

First, organize the data in numerical order.

6, 6, 6, 6, 6, 7, 8, 8, 8, 8, 11, 11, 12, 12, 12, 13, 14

Next, create a frequency table. There are two columns in the frequency table. The first column shows the number of ducks and the second column shows how many times that number of ducks was on the pond.

**Table 10.9:**

<table>
<thead>
<tr>
<th>Number of Ducks</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>
### Table 10.9: (continued)

<table>
<thead>
<tr>
<th>Number of Ducks</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
</tr>
</tbody>
</table>

Then, turn the frequency table into a line plot.

```
    x
    x  x
    x  x  x
    x  x  x  x  x
    x  x  x  x  x  x  x  x
```

Here are some conclusions that can be drawn by looking at both methods of displaying data:

- In both, the range of numbers is shown. There were between 6 and 14 ducks seen, so each number from 6 to 14 is represented.
- There weren’t any days where 9 or 10 ducks were counted, yet both are represented because they fall in the range of ducks counted.
- Both methods help to visually understand data and its meaning.

### Use the data below, which shows the enrollment in different college classes, to answer the following questions.

```
    x
    x  x
    x  x  x  x
    x  x  x  x  x  x
    x  x  x  x  x  x  x
```

**Example 3**

How many classes at the college have 31 students in them?

First, find the data from the question on the line plot.
Next, look at the frequency represented by X’s to see how many classes had 31 students. Then, answer the question with the data that is shown.
The answer is 3 classes have 31 students in them.

**Example 4**

How many classes at the college have 38 students in them?
First, find the data from the question on the line plot.
Next, look at the frequency represented by X’s to see how many classes have 38 students.
Then, answer the question with the data shown.
The answer is 1 class has 38 students in it.

**Example 5**

How many classes at the college have 33 students in them?
First, find the data from the question on the line plot.
Next, look at the frequency represented by X’s to see how many classes have 33 students.
Then, answer the question with the data shown.
The answer is 4 classes have 33 students in them.

**Review**

Here is a line plot that shows how many seals came into the harbor in La Jolla California during an entire month. Use it to answer the following questions.

<table>
<thead>
<tr>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

1. How many times did thirty seals appear on the beach?
2. Which two categories have the same frequency?
3. How many times were 50 or more seals counted on the beach?
4. True or False. This line plot shows us the number of seals that came on each day of the month.
5. True or False. There weren’t any days that less than 30 seals appeared on the beach.
6. How many times were 60 seals on the beach?
7. How many times were 70 seals on the beach?
8. What is the smallest number of seals that was counted on the beach?
9. What is the greatest number of seals that were counted on the beach?
10. Does the frequency table show any number of seals that weren’t counted at all?

Organize each list of data. Then create a frequency table to show the results. There are two answers for each question.
11. 8, 8, 2, 2, 2, 2, 5, 6, 3, 3, 4
12. 20, 18, 18, 19, 19, 17, 17, 17, 17, 17
13. 100, 99, 98, 92, 92, 92, 92, 92, 98, 98
14. 75, 75, 75, 70, 70, 70, 71, 72, 72, 74, 74, 74
15. 1, 1, 1, 1, 2, 2, 2, 3, 3, 5, 5, 5, 5, 5, 5, 5

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.10.

Resources
10.5. Line Graphs to Display Data Over Time

In this concept, you will learn how to create a line graph and analyze data displayed on a line graph.

Line Graphs

A line graph is a graph that shows how data changes over time.

To make a line graph, you need a collection of data that has changed over time. Data that shows growth over years is a good example of appropriate data for a line graph.
When Jamal was born, his parents planted a tree in the back yard. Here is how tall the tree was in each of the next five years.

2003- 2 ft.
2004- 3 ft.
2005- 5 ft.
2006- 9 ft.
2007- 14 ft.

Like other graphs, line graphs need two axes, one vertical (Y-axis) and one horizontal (X-axis).

The vertical axis represents the range of tree growth. The tree grew from 2 feet to 14 feet. That is the scale. The horizontal axis represents the years when tree growth was calculated.

Here is a finished line graph representing this data:

![Tree Growth from 2003-2007](image)

It is important to know how to create line graphs for the purpose of reading and analyzing the data displayed in the graph. Patterns and trends can be concluded by reading line graphs to answer questions. In the Guided Practice section, you will be practicing answer specific questions related to a new set of data.

**Examples**

**Example 1**

Earlier, you were given a problem about Jane and Spot.

Jane wanted to track Spot’s growth from 1 month old to 9 months old. She kept track of his weight gain and recorded the following data:

1 month- 8 lb.
2 months- 13 lb.
3 months- 17 lb.
4 months- 24 lb.
5 months- 27 lb.
10.5. Line Graphs to Display Data Over Time

Jane wants to graph this data and then figure out Spot’s total weight gain from 1 month to 9 months.

First, Jane creates her line graph. She creates an x axis to represent months and a y axis to represent weight in pounds. She plots her points on the graph and connects them to create a line in the graph.

Next, Jane begins to figure out the difference in weight from 1 month old to 9 months old. She looks for Spot’s weight at 1 month first. His weight was about 8 lb.

Then, Jane finds the data showing his weight at 9 months old. Spot weighs about 34 lb. now.

To find the total weight gain, Jane finds the difference between these two data points by subtracting.

\[34 - 8 = 26\]

The answer is Spot gained a total of 26 pounds from when he was a puppy until he was fully grown at 9 months.

Example 2

Consider the following line graph.
Which day of the week had the highest temperature? What was that temperature?

First, analyze the graph to find the highest point in the trend line.

Next, follow down to the X-axis from that highest point to see what time that data represents. The highest point in this graph is data from February 3rd.

Then, go back to the highest point and follow across to the Y-axis for the value that is represented by that point on the graph. This point falls between 30 and 35 degrees somewhere in the middle. So a good estimate is about 33 degrees.

The answer is February 3rd. The temperature on that day was about 33 degrees.

Use the data from the graph "This Week’s Temperatures in Boston" to answer the following questions.
Example 3

What was the temperature in Boston on February 1st?
First, find the data for February 1st on the X-axis.
Next, follow up the graph to the data point for that day.
Then, follow across to the Y-axis to determine the temperature on that day. Some estimating may be necessary if the point does not fall directly on a number but more in between two data ranges.
The answer is about 27 degrees.

Example 4

What day that week had the lowest temperature in Boston?
First, look at the line in the graph to find the lowest data point on the line. In this case, there are two low points that seem to be the same temperature.
Next, follow down to the X-axis to determine the day the lowest point occurred.
Then, do the same with the other low point in the line to see the other day that had the same low temperature.
The answer is that both February 5th and February 6th had the lowest temperature that week.

Example 5

What is the greatest difference in temperature in Boston that week?
First, gather the data for the highest temperature that week. The high temperature was about 33 degrees on February 3rd.
Next, gather the data for the lowest temperature that week so you can find the difference. The lowest temperature that week was about 26 degrees on two different days (February 5th and 6th).
Then, find the difference between the high and low temperatures that week by subtracting.

\[33 - 26 = 7\]

The answer is there was a difference of 7 degrees.

**Review**

Use the following line graph to answer each question.

1. What is being measured in this line graph?
2. What is on the horizontal axis?
3. What is on the vertical axis?
4. What was the highest temperature recorded?
5. What was the lowest temperature recorded?
6. What is the difference between the two temperatures?
7. On what day did the lowest temperature occur?
8. What was the average temperature for the week?
9. What was the median temperature for the week?
10. Did any two days have the same temperature?
11. What was that temperature?
12. On which two days did it occur?
13. Based on this trend, would the temperature on February 8th be less than 30 degrees or greater than?
14. True or false. There isn’t a way to figure out the temperature on January 31st.
15. What was the temperature on February 5th?

**Review (Answers)**

To see the Review answers, open this PDF file and look for section 2.15.
Here you’ll learn how to construct and interpret box-and-whisker plots.

**Box-and-Whisker Plots**

In traditional statistics, data is organized by using a frequency distribution. The results of the frequency distribution can then be used to create various graphs, such as a histogram or a frequency polygon, which indicate the shape or nature of the distribution. The shape of the distribution will allow you to confirm various conjectures about the nature of the data.

To examine data in order to identify patterns, trends, or relationships, exploratory data analysis is used. In exploratory data analysis, organized data is displayed in order to make decisions or suggestions regarding further actions. A box-and-whisker plot (often called a box plot) can be used to graphically represent the data set, and the graph involves plotting 5 specific values. The 5 specific values are often referred to as a five-number summary of the organized data set. The five-number summary consists of the following:

1. The lowest number in the data set (minimum value)
2. The median of the lower quartile: $Q_1$ (median of the first half of the data set)
3. The median of the entire data set (median)
4. The median of the upper quartile: $Q_3$ (median of the second half of the data set)
5. The highest number in the data set (maximum value)

The display of the five-number summary produces a box-and-whisker plot as shown below:

The above model of a box-and-whisker plot shows 2 horizontal lines (the whiskers) that each contain 25% of the data and are of the same length. In addition, it shows that the median of the data set is in the middle of the box,
which contains 50% of the data. The lengths of the whiskers and the location of the median with respect to the center of the box are used to describe the distribution of the data. It’s important to note that this is just an example. Not all box-and-whisker plots have the median in the middle of the box and whiskers of the same size.

Information about the data set that can be determined from the box-and-whisker plot with respect to the location of the median includes the following:

a. If the median is located in the center or near the center of the box, the distribution is approximately symmetric.
b. If the median is located to the left of the center of the box, the distribution is positively skewed.
c. If the median is located to the right of the center of the box, the distribution is negatively skewed.

Information about the data set that can be determined from the box-and-whisker plot with respect to the length of the whiskers includes the following:

a. If the whiskers are the same or almost the same length, the distribution is approximately symmetric.
b. If the right whisker is longer than the left whisker, the distribution is positively skewed.
c. If the left whisker is longer than the right whisker, the distribution is negatively skewed.

The length of the whiskers also gives you information about how spread out the data is.

A box-and-whisker plot is often used when the number of data values is large. The center of the distribution, the nature of the distribution, and the range of the data are very obvious from the graph. The five-number summary divides the data into quarters by use of the medians of the upper and lower halves of the data. Many data sets contain values that are either extremely high values or extremely low values compared to the rest of the data values. These values are called outliers. There are several reasons why a data set may contain an outlier. Some of these are listed below:

1. The value may be the result of an error made in measurement or in observation. The researcher may have measured the variable incorrectly.
2. The value may simply be an error made by the researcher in recording the value. The value may have been written or typed incorrectly.
3. The value could be a result obtained from a subject not within the defined population. A researcher recording marks from a math 12 examination may have recorded a mark by a student in grade 11 who was taking math 12.
4. The value could be one that is legitimate but is extreme compared to the other values in the data set. (This rarely occurs, but it is a possibility.)

If an outlier is present because of an error in measurement, observation, or recording, then either the error should be corrected, or the outlier should be omitted from the data set. If the outlier is a legitimate value, then the statistician must make a decision as to whether or not to include it in the set of data values. There is no rule that tells you what to do with an outlier in this case.

One method for checking a data set for the presence of an outlier is to follow the procedure below:

1. Organize the given data set and determine the values of \( Q_1 \) and \( Q_3 \).
2. Calculate the difference between \( Q_1 \) and \( Q_3 \). This difference is called the interquartile range (IQR): \( IQR = Q_3 - Q_1 \).
3. Multiply the difference by 1.5, subtract this result from \( Q_1 \), and add it to \( Q_3 \).
4. The results from Step 3 will be the range into which all values of the data set should fit. Any values that are below or above this range are considered outliers.
Listing a Five-Number Summary and Describing the Distribution

For each box-and-whisker plot, list the five-number summary and describe the distribution based on the location of the median.

a. Minimum value → 4  
   $Q_1 \rightarrow 6$  
   Median → 9  
   $Q_3 \rightarrow 10$  
   Maximum value → 12  
   The median of the data set is located to the right of the center of the box, which indicates that the distribution is negatively skewed.

b. Minimum value → 225  
   $Q_1 \rightarrow 250$  
   Median → 300  
   $Q_3 \rightarrow 325$  
   Maximum value → 350  
   The median of the data set is located to the right of the center of the box, which indicates that the distribution is negatively skewed.

c. Minimum value → 60  
   $Q_1 \rightarrow 70$
Median → 75
$Q_3$ → 95
Maximum value → 100
The median of the data set is located to the left of the center of the box, which indicates that the distribution is positively skewed.

**Constructing a Box-and-Whisker Plot**

The numbers of square feet (in 100s) of 10 of the largest museums in the world are shown below:
650, 547, 204, 213, 343, 288, 222, 250, 287, 269

Construct a box-and-whisker plot for the above data set and describe the distribution.

The first step is to organize the data values as follows:

20,400 21,300 22,200 25,000 26,900 28,700 28,800 34,300 54,700 65,000

Now calculate the median, $Q_1$, and $Q_3$.

Median → \( \frac{26,900 + 28,700}{2} = \frac{55,600}{2} = 27,800 \)

$Q_1$ = 22,200
$Q_3$ = 34,300
Next, complete the following list:
Minimum value → 20,400
$Q_1$ → 22,200
Median → 27,800
$Q_3$ → 34,300
Maximum value → 65,000

The right whisker is longer than the left whisker, which indicates that the distribution is positively skewed.
Checking for Outliers

Using the procedure outlined above, check the following data sets for outliers:

a. 18, 20, 24, 21, 5, 23, 19, 22
Organize the given data set as follows:

18, 20, 24, 21, 5, 23, 19, 22
5, 18, 19, 20, 21, 22, 23, 24

Determine the values for $Q_1$ and $Q_3$.

$Q_1 = \frac{18 + 19}{2} = \frac{37}{2} = 18.5$
$Q_3 = \frac{22 + 23}{2} = \frac{45}{2} = 22.5$

Calculate the difference between $Q_1$ and $Q_3$: $Q_3 - Q_1 = 22.5 - 18.5 = 4.0$.
Multiply this difference by 1.5: $(4.0)(1.5) = 6.0$.
Finally, compute the range.

$Q_1 - 6.0 = 18.5 - 6.0 = 12.5$

$Q_3 + 6.0 = 22.5 + 6.0 = 28.5$.

Are there any data values below 12.5? Yes, the value of 5 is below 12.5 and is, therefore, an outlier.
Are there any values above 28.5? No, there are no values above 28.5.

b. 12, 15, 19, 14, 26, 17, 12, 42, 18
Organize the given data set as follows:

13, 15, 19, 14, 26, 17, 12, 42, 18
12, 13, 14, 15, 17, 18, 19, 26, 42

Determine the values for $Q_1$ and $Q_3$.

12, [13, 14], 15, [17], 18, [19, 26], 42
$Q_1 = \frac{13 + 14}{2} = \frac{27}{2} = 13.5 \quad Q_3 = \frac{19 + 26}{2} = \frac{45}{2} = 22.5$

Calculate the difference between $Q_1$ and $Q_3$: $Q_3 - Q_1 = 22.5 - 13.5 = 9.0$.
Multiply this difference by 1.5: $(9.0)(1.5) = 13.5$.
Finally, compute the range.

$Q_1 - 13.5 = 13.5 - 13.5 = 0$

$Q_3 + 13.5 = 22.5 + 13.5 = 36.0$

Are there any data values below 0? No, there are no values below 0.
Are there any values above 36.0? Yes, the value of 42 is above 36.0 and is, therefore, an outlier.

**Points to Consider**

- Are there still other ways to represent data graphically?
- Are there other uses for a box-and-whisker plot?
- Can box-and-whisker plots be used for comparing data sets?

**Examples**

For the following data sets, determine the five-number summaries:

**Example 1**

12, 16, 36, 10, 31, 23, 58
The first step is to organize the values in the data set as shown below:

12, 16, 36, 10, 31, 23, 58
10, 12, 16, 23, 31, 36, 58
10.6. Box-and-Whisker Plots

Now complete the following list:
Minimum value → 10
$Q_1$ → 12
Median → 23
$Q_3$ → 36
Maximum value → 58

**Example 2**

144, 240, 153, 629, 540, 300

The first step is to organize the values in the data set as shown below:

144, 240, 153, 629, 540, 300
144, 153, 240, 300, 540, 629

Now complete the following list:
Minimum value → 144
$Q_1$ → 153
Median → 270
$Q_3$ → 540
Maximum value → 629

**Example 3**

Use the data set from Example 1 and the five-number summary to construct a box-and-whisker plot to model the data set.

The five-number summary can now be used to construct a box-and-whisker plot for part i. Be sure to provide a scale on the number line that includes the range from the minimum value to the maximum value.
Minimum value $\rightarrow 10$

$Q_1 \rightarrow 12$

Median $\rightarrow 23$

$Q_3 \rightarrow 36$

Maximum value $\rightarrow 58$

It is very visible that the right whisker is much longer than the left whisker. This indicates that the distribution is positively skewed.

**Review**

1. Which of the following is not a part of the five-number summary?
   a. $Q_1$ and $Q_3$
   b. the mean
   c. the median
   d. minimum and maximum values

2. What percent of the data is contained in the box of a box-and-whisker plot?
   a. 25%
   b. 100%
   c. 50%
   d. 75%

3. What name is given to the horizontal lines to the left and right of the box of a box-and-whisker plot?
   a. axis
   b. whisker
   c. range
   d. plane

4. What term describes the distribution of a data set if the median of the data set is located to the left of the center of the box in a box-and-whisker plot?
   a. positively skewed
   b. negatively skewed
   c. approximately symmetric
   d. not skewed

5. What 2 values of the five-number summary are connected with 2 horizontal lines on a box-and-whisker plot?
   a. Minimum value and the median
   b. Maximum value and the median
   c. Minimum and maximum values
   d. $Q_1$ and $Q_3$

6. For the following data sets, determine the five-number summaries:
   a. 74, 69, 83, 79, 60, 75, 67, 71
   b. 6, 9, 3, 12, 11, 9, 15, 5, 7
7. For each of the following box-and-whisker plots, list the five-number summary and comment on the distribution of the data:

![Box-and-Whisker Plot]

a. 

b. 

8. The following data represents the number of coins that 12 randomly selected people had in their piggy banks:

35 58 29 44 104 39 72 34 50 41 64 54

Construct a box-and-whisker plot for the above data.

9. The following data represent the time (in minutes) that each of 20 people waited in line at a local book store to purchase the latest Harry Potter book:

15 8 5 10 14 17 21 23 6 19 31 34 30 31 3 22 17 25 5 16

Construct a box-and-whisker plot for the above data. Are the data skewed in any direction?

10. Firman’s Fitness Factory is a new gym that offers reasonably-priced family packages. The following table represents the number of family packages sold during the opening month:

24 21 31 28 29
27 22 27 30 32
26 35 24 22 34
30 28 24 32 27
32 28 27 32 23
20 32 28 32 34

Construct a box-and-whisker plot for the data. Are the data symmetric or skewed?

11. Shown below is the number of new stage shows that appeared in Las Vegas for each of the past several years. Construct a box-and-whisker plot for the data and comment of the shape of the distribution.

31 29 34 30 38 40 36 38 32 39 35

12. The following data represent the average snowfall (in centimeters) for 18 Canadian cities for the month of January. Construct a box-and-whisker plot to model the data. Is the data skewed? Justify your answer.

<table>
<thead>
<tr>
<th>Name of City</th>
<th>Amount of Snow(cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calgary</td>
<td>123.4</td>
</tr>
<tr>
<td>Charlottetown</td>
<td>74.5</td>
</tr>
<tr>
<td>Edmonton</td>
<td>80.6</td>
</tr>
<tr>
<td>Fredericton</td>
<td>73.8</td>
</tr>
<tr>
<td>Halifax</td>
<td>64.0</td>
</tr>
<tr>
<td>Labrador City</td>
<td>110.4</td>
</tr>
<tr>
<td>Moncton</td>
<td>82.4</td>
</tr>
<tr>
<td>Montreal</td>
<td>63.6</td>
</tr>
</tbody>
</table>
TABLE 10.10: (continued)

<table>
<thead>
<tr>
<th>Name of City</th>
<th>Amount of Snow (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ottawa</td>
<td>48.9</td>
</tr>
<tr>
<td>Quebec City</td>
<td>53.8</td>
</tr>
<tr>
<td>Regina</td>
<td>35.9</td>
</tr>
<tr>
<td>Saskatoon</td>
<td>25.4</td>
</tr>
<tr>
<td>St. John’s</td>
<td>97.5</td>
</tr>
<tr>
<td>Sydney</td>
<td>44.2</td>
</tr>
<tr>
<td>Toronto</td>
<td>21.8</td>
</tr>
<tr>
<td>Vancouver</td>
<td>12.8</td>
</tr>
<tr>
<td>Victoria</td>
<td>8.3</td>
</tr>
<tr>
<td>Winnipeg</td>
<td>76.2</td>
</tr>
</tbody>
</table>

13. Using the procedure outlined in this concept, check the following data sets for outliers:
   a. 25, 33, 55, 32, 17, 19, 15, 18, 21
   b. 149, 123, 126, 122, 129, 120

**Review (Answers)**

To view the Review answers, open this PDF file and look for section 7.11.
Here you’ll learn how to use Texas Instruments calculators to create box-and-whisker plots and to determine the five-number summary of a data set.

**Applications of Box-and-Whisker Plots**

The TI-83 or TI-84 can be used to create a box-and-whisker plot. In the following examples, the TI-83 is used. In later Concepts, key strokes using the TI-84 will be presented to you. The five-number summary values can be determined by using the TRACE feature and the left and right arrows of the calculator or by pressing

![TRACE feature](https://www.ck12.org/flx/render/embeddedobject/137050)

and using 1-Var Stats on the CALC menu. The **TRACE feature** gives the individual values of the five-number summary one at a time.

**Using Technology to Construct a Box-and-Whisker Plot**

The following numbers represent the number of siblings in each family for 15 randomly selected students:

4, 1, 2, 2, 5, 3, 4, 2, 6, 4, 6, 1, 7, 8, 4

Use technology to construct a box-and-whisker plot to display the data.

A box-and-whisker plot can be created with a TI-83 calculator as shown below:
Note that when creating a box-and-whisker plot with a TI calculator, you don’t have to actually sort the data. The calculator will sort the data automatically when creating the box-and-whisker plot.

**Listing the Five-Number Summary**

List the five-number summary values for the data in the previous example.

The five-number summary can be obtained from the calculator in 2 ways.

a. The following results are obtained by simply using the TRACE feature and the left and right arrows:

The values at the bottom of each screen are the five-number summary.

b. The second method involves pressing

and using 1-Var Stats on the CALC menu for L1:
Constructing Box-and-Whisker Plots

Construct a box-and-whisker plot using technology to represent the average number of sick days used by 9 employees of a large industrial plant. The numbers of sick days are as follows:

39  31  18  34  25  22  32  23  22

What are the values for the five-number summary?

The screens below show the five-number summary:

The values for the five-number summary are as follows:

Min. Value → 18

$Q_1$ → 22

Med → 25

$Q_3$ → 33

Max. Value → 39

This can be verified by pressing

and using 1-Var Stats on the CALC menu for L1.

1-Var Stats

$\hat{n}=9$

$\min X=18$

$Q_1=22$

$\text{Med}=25$

$Q_3=33$

$\max X=39$
Example

Example 1

The following data represents the number of flat-screen televisions assembled at a local electronics company for a sample of 28 days:

48 55 51 44 59 49 47
45 51 56 50 57 53 55
47 49 51 54 56 54 47
50 53 52 55 51 59 48

Using technology, construct a box-and-whisker plot for the data. What are the values for the five-number summary?

First press

STAT

and choose Edit on the EDIT menu. Enter the data values into L1, and then press

2ND

Y=

. Press

ENTER

and make sure that your settings are as follows. Note that for Type, the middle graph in the second row is selected.

Now press

WINDOW

and make sure that your settings are as shown below:
Finally, press \( \text{GRAPH} \). The following screens show the steps necessary to determine the five-number summary using the TI-83:

The values for the five-number summary are as shown below:

Min. Value \( \rightarrow 44 \)

\( Q_1 \rightarrow 48.5 \)

Med \( \rightarrow 51 \)

\( Q_3 \rightarrow 55 \)

Max. Value \( \rightarrow 59 \)

This can be verified by pressing \( \text{STAT} \) and using 1-Var Stats on the CALC menu for L1.
For each of the following box-and-whisker plots, determine the five-number summary and give one possible data set that could produce the box-and-whisker plot.
10.7. Applications of Box-and-Whisker Plots

3.

4.

5.

6.
Review (Answers)

To view the Review answers, open this PDF file and look for section 7.12.
10.8 Represent Real-World Data Using Bar Graphs, Frequency Tables and Histograms

Here you’ll learn to represent real-world data by using bar graphs, frequency tables and histograms.

The students at Smith Middle School returned to classes and found out that there was a big surprise. Wood shop had been cancelled from the curriculum.

“How could they have cancelled wood shop?” Kyle asked when looking at his schedule. “I have been waiting all year for this.”

In the past, only the students in the seventh and eighth grades were allowed to take wood shop. Many students spent all of sixth grade waiting for wood shop.

“I know, I was waiting for it too,” Sarah said.

“There must be a mistake,” Tanisha commented.

However, there wasn’t a mistake. The administration was sure that there wasn’t enough student interest to continue having wood shop.

This set the students in the seventh and eighth grade on a mission. That very first day after school, the students held their own meeting on the football field. They decided to gather data and prove to the administration that wood shop was a necessary part of the curriculum.

Over the next few weeks, the students worked hard to gather data about wood shop. They learned that in 2008, there were 30 out of 100 seventh graders and 40 out of 100 eighth graders who had participated in wood shop. Then in 2009, the numbers had increased. There were 40 seventh graders and 58 eighth graders who had participated.

“This is great!” Kyle said. “Now we can prove that wood shop is wanted!”

“Yes, but I think we should draw a chart to show our results,” Tanisha said.

This is where you come in. In this Concept, you will learn how to draw different types of graphs to display data. Pay close attention because at the end of the Concept you will need to draw a bar graph to show the data that was collected.

Guidance

Real-world data can be easily and accurately represented by using bar graphs, frequency tables and histograms. Let’s look at how to work with these data displays.
The data table below depicts the ages of twenty of our nation’s presidents at the time of Inauguration. Create a bar graph, frequency table, and histogram to display the data.

<table>
<thead>
<tr>
<th>President</th>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>George Washington</td>
<td>57</td>
</tr>
<tr>
<td>John Adams</td>
<td>61</td>
</tr>
<tr>
<td>Thomas Jefferson</td>
<td>57</td>
</tr>
<tr>
<td>James Madison</td>
<td>57</td>
</tr>
<tr>
<td>James Monroe</td>
<td>58</td>
</tr>
<tr>
<td>John Quincy Adams</td>
<td>57</td>
</tr>
<tr>
<td>Andrew Jackson</td>
<td>61</td>
</tr>
<tr>
<td>Martin Van Buren</td>
<td>54</td>
</tr>
<tr>
<td>William Henry Harrison</td>
<td>68</td>
</tr>
<tr>
<td>John Tyler</td>
<td>51</td>
</tr>
<tr>
<td>Dwight D. Eisenhower</td>
<td>62</td>
</tr>
<tr>
<td>John F. Kennedy</td>
<td>43</td>
</tr>
<tr>
<td>Lyndon B. Johnson</td>
<td>55</td>
</tr>
<tr>
<td>Richard Nixon</td>
<td>56</td>
</tr>
<tr>
<td>Gerald Ford</td>
<td>61</td>
</tr>
<tr>
<td>Jimmy Carter</td>
<td>52</td>
</tr>
<tr>
<td>Ronald Regan</td>
<td>69</td>
</tr>
<tr>
<td>Bill Clinton</td>
<td>46</td>
</tr>
</tbody>
</table>

To create a bar graph, first draw a horizontal \((x)\) and vertical \((y)\) axis.

Label the horizontal axis with each president’s last name.

Label the vertical axis with intervals of two, beginning with the number thirty.

Next, draw a vertical column to the appropriate value for each president.
Now, create a frequency table by drawing three columns. Designate the first column for intervals of four. The middle column is to tally the ages. The final column depicts the total frequency for each interval.

<table>
<thead>
<tr>
<th>Ages</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>43 - 46</td>
<td></td>
<td></td>
</tr>
<tr>
<td>47 - 50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>51 - 54</td>
<td></td>
<td></td>
</tr>
<tr>
<td>55 - 58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>59 - 62</td>
<td></td>
<td></td>
</tr>
<tr>
<td>63 - 66</td>
<td></td>
<td></td>
</tr>
<tr>
<td>67 - 70</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Finally, create the histogram. To create a histogram, first draw a horizontal \( (x) \) and vertical \( (y) \) axis. Label the horizontal axis with the intervals depicted on the frequency table. Label the vertical axis by ones. Draw a vertical column to the appropriate value for each interval on the horizontal axis. Recall that there is no space between the vertical columns on a histogram.
Examine the bar graph to answer the following questions about our presidents.

**Example A**

How many of our presidents were sixty-one years of age or older at the time of the Inauguration?

**Solution:** Seven presidents
Example B

What was the most popular age for a president at the time of the Inauguration?

Solution: Fifty-seven years old

Example C

What was the age of the youngest president? Who was it?

Solution: Forty-three years old, Kennedy

Now let’s go back to the dilemma at the beginning of the Concept.

It is time to draw a bar graph to represent the data. Remember that the bar graph that you draw needs to show the data for 2008 and 2009.

To create a bar graph, remember you will need a vertical axis and a horizontal axis.

Because there are 100 students in both the seventh and the eighth grade, you will be able to figure out the intervals for the vertical axis. The horizontal axis should show the numbers of seventh and eighth graders for both 2008 and 2009.

Here is a bar graph to show the data.

This bar graph will help the students to show the administration how popular wood shop has become!

Guided Practice

Here is one for you to try on your own.

Use this histogram and figure out how many students spend between 1 hour and 1 1/2 hours studying.
10.8. Represent Real-World Data Using Bar Graphs, Frequency Tables and Histograms

Solution

If you look at the histogram, you can see that the horizontal axis represents the hours that students spend studying. The vertical axis represents the number of students who study for each period of time.

Given this histogram, there are 35 students who spend between 1 hour and 1 1/2 hours studying.

Video Review

Khan Academy Histograms

Explore More

Directions: Use the histogram from the Guided Practice to answer the following questions.
1. How many students spend less than 30 minutes studying?
2. How many students spend greater than 60 minutes studying?
3. How many students spend greater than 90 minutes studying?
4. How many students spend between 2 and 2 1/2 hours studying?
5. If this histogram is the result of a survey, was the number of students surveyed greater than 100?

**Directions:** Use this bar graph to answer the following questions.

6. If each girl could only vote once, what was the total number of girls surveyed?
7. If each boy could only vote once, how many boys were surveyed?
8. What fraction of the girls surveyed chose track as their favorite sport?
9. What fraction of the girls surveyed chose soccer as their favorite sport?
10. What percentage of the girls surveyed chose track and soccer as their favorite sport? You may round to the nearest whole percent.
11. What fraction of the boys chose football as their favorite sport?
12. How many boys and girls were surveyed in all?
13. True or false. Basketball is the least popular sport among girls.
14. True or false. It is also the least popular among boys.
15. What is the most popular sport among girls?
16. What is the least popular sport among boys?
In this concept, you will learn to use frequency tables and histograms to display data.

The coach of the Markwell Cougars track team wants to compare the heights of his team to that of their rivals, the Sampson Hawks. He was wondering if there is a correlation between speed and height.

The coach wrote the following heights from smallest to largest.

Markwell Cougars:
170, 172, 175, 176, 176, 178, 181, 182, 183, 183, 183, 185, 185, 187, 188, 188, 189, 190, 195

Sampson Hawks:

Create a visual display of this data.

In this concept, you will learn to use frequency tables and histograms.

**Frequency**

A visual display is used to show data. Each type of visual tool has advantages and the best type of plot or graph depends on the situation. Indeed, sometimes it is a matter of preference as many different graphs could be used to illustrate the same data.

Let’s take a look at frequency tables and histograms.

**Frequency** is a measure of how often something occurs. A frequency table is used to measure and visually show how often a data value occurs.
Let’s look at an example.

A teacher is preparing for parent conferences. In order to provide parents with the most information possible about their children, he wants to organize the grades of the class so that they can compare the grades to the rest of the class.

The math percentages have been calculated and his students earned the following grades:

88, 86, 92, 65, 72, 75, 81, 84, 85, 93, 99, 50, 78, 80, 86, 76, 74, 95, 81, 87, 90, 72, 76, 61, 85, 84, 78, 83.

Grades are determined by percent where 0-59% is an F, 60-69% is a D, 70-79% is a C, 80-89% is a B, and 90-100% is an A. These values make the most logical intervals. **Intervals** are always chosen depending on the range of the data. He will make a frequency table to illustrate the information.

First, for each student who scored in the given range, he puts an X. Sometimes, frequency tables use X’s and other times, they can use lines for tally marks.

<table>
<thead>
<tr>
<th>Table 10.11: Frequency Tables of Grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval</td>
</tr>
<tr>
<td>90 - 100</td>
</tr>
<tr>
<td>80 - 89</td>
</tr>
<tr>
<td>70 - 79</td>
</tr>
<tr>
<td>60 - 69</td>
</tr>
<tr>
<td>0 - 59</td>
</tr>
</tbody>
</table>

This tally is useful in the sense that it communicates to parents how many students in the class scored in the A range, B range, etc. It would not be as important for the parents to see the individual scores of each student as it would be to see the total number of students in each interval. That way, if their child earned a B, then they would know that the child falls in a category that most other students scored in. If a child earned a D, for example, it would indicate that they are below the general level of the other students and might need additional help.

Next, he creates a **histogram**. A **histogram** is similar to a bar graph in that it uses columns to illustrate data on x- and y-axes. In a histogram, you can use the same intervals as you did for the frequency table. The bars in the histogram will have no space between them.
The histogram shows the same information as the frequency table does. However, the histogram is a type of graph, meaning that it is visual representation. The bars on the histogram are interpreted more easily by size than numerical data.

**Examples**

**Example 1**

Earlier, you were given a problem about the track teams and the heights of the runners.

The coach is comparing the heights of his team, the Markwell Cougars to the rival team, the Sampson Hawks. You need to create a double histogram for the data. The data collected is:

Markwell Cougars:
170, 172, 175, 176, 176, 178, 181, 182, 183, 183, 185, 185, 187, 188, 188, 189, 190, 195

Sampson Hawks:

First, create a frequency table for the data. Since you are comparing two sets of data, you have to put both data sets into the frequency table.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Markwell Cougars</th>
<th>Sampson Hawks</th>
</tr>
</thead>
<tbody>
<tr>
<td>160 - 169</td>
<td></td>
<td></td>
</tr>
<tr>
<td>170 - 179</td>
<td></td>
<td></td>
</tr>
<tr>
<td>180 - 189</td>
<td></td>
<td></td>
</tr>
<tr>
<td>190 - 199</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Next, use this data to create a histogram that compares the data.
You can see from the histogram that both teams have more players in the 180 - 189 interval. However, while the Cougars have more players in the 170 - 179 interval, the Hawks have slightly more in the taller interval. The Hawks have a slight height advantage.

**Example 2**

Create a histogram of the mass of geodes found at a volcanic site. Scientists measured 24 geodes in kilograms and got the following data:

0.8, 0.9, 1.1, 1.1, 1.2, 1.5, 1.5, 1.6, 1.7, 1.7, 1.7, 1.9, 2.0, 2.3, 5.3, 6.8, 7.5, 9.6, 10.5, 11.2, 12.0, 17.6, 23.9, and 26.8.

First, let’s think about intervals.

The minimum item is 0.8 kg and the maximum is 26.8. To get a good idea of the data, you could use intervals that encompass perhaps 4 kg intervals, 5 kg intervals, or 6 kg intervals. Let’s try intervals of 5 kg.

Begin with a frequency table.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 5</td>
<td>@@@@</td>
<td>14</td>
</tr>
<tr>
<td>5.1 - 10</td>
<td>@@@</td>
<td>4</td>
</tr>
<tr>
<td>10.1 - 15</td>
<td>@@@</td>
<td>3</td>
</tr>
<tr>
<td>15.1 - 20</td>
<td>@</td>
<td>1</td>
</tr>
<tr>
<td>20.1 - 25</td>
<td>@</td>
<td>1</td>
</tr>
<tr>
<td>25.1 - 30</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

Next, create a histogram for this data.
Example 3

True or false: An interval is the frequency that an event happens.
The answer is false.
Intervals are always chosen depending on the range of the data.

Example 4

True or false: To create a histogram, you first need a frequency table.
The answer is true.

Example 5

True or false: If I wanted to create a histogram on the number of people who went to the town movie theater on the weekend, I would first need to figure out how many people went to the movies on each weekend day and night.
The answer is true.
The data collected would help create a frequency table and then you could create your histogram.

Review

Use what you have learned about histograms to answer each question.

**Table 10.14:** Hours Slept Each Night

<table>
<thead>
<tr>
<th>Number of Hours Slept</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>I</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>II</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>IIIII</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>IIII</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>IIII</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>IIII</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>II</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>II</td>
<td>2</td>
</tr>
</tbody>
</table>

1. Create a histogram that illustrates this data.
2. Explain why you chose the intervals that you chose.
3. What can you interpret from your histogram?

Compare the stem-and-leaf plot to the histogram of Melanie’s Christmas gift expenses. She told her husband, “Most of the gifts were about $60.”

4. Is she telling the truth?
5. Which tool is more useful in making a decision about her truthfulness?
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

6. Looking at this histogram, can you conclude that most people exercise between 6 - 11 hours per week?
7. What is the fewest number of hours?
8. What is the range of hours?
9. Why do you think they chose the interval that they did?
Conduct your own survey and collect data. Choose attendance rates in your class or vacation days per year for example. Then create a frequency table, histogram and analyze your data. Explain why you chose the interval that you did and which data set had the greatest and least results.

**Review (Answers)**

To see the Review answers, open this [PDF file](#) and look for section 10.4.
Here you’ll learn how to create a frequency distribution chart and how to construct and interpret a histogram.

**Histograms**

An extension of the bar graph is the histogram. A histogram is a type of vertical bar graph in which the bars represent grouped continuous data. The shape of a histogram can tell you a lot about the distribution of the data, as well as provide you with information about the mean, median, and mode of the data set. The following are some typical histograms, with a caption below each one explaining the distribution of the data, as well as the characteristics of the mean, median, and mode. Distributions can have other shapes besides the ones shown below, but these represent the most common ones that you will see when analyzing data. In each of the graphs below, the distributions are not perfectly shaped, but are shaped enough to identify an overall pattern.

a)

![Bell-Shaped Unimodal Distribution](image)

Figure a represents a bell-shaped distribution, which has a single peak and tapers off to both the left and to the right of the peak. The shape appears to be symmetric about the center of the histogram. The single peak indicates that the distribution is unimodal. The highest peak of the histogram represents the location of the mode of the data set. The mode is the data value that occurs the most often in a data set. For a symmetric histogram, the values of the mean, median, and mode are all the same and are all located at the center of the distribution.

b)
c) Figure b represents a distribution that is approximately uniform and forms a rectangular, flat shape. The frequency of each class is approximately the same.

d) Figure c represents a right-skewed distribution, which has a peak to the left of the distribution and data values that taper off to the right. This distribution has a single peak and is also unimodal. For a histogram that is skewed to the right, the mean is located to the right on the distribution and is the largest value of the measures of central tendency. The mean has the largest value because it is strongly affected by the outliers on the right tail that pull the mean to the right. The mode is the smallest value, and it is located to the left on the distribution. The mode always occurs at the highest point of the peak. The median is located between the mode and the mean.
Figure d represents a **left-skewed distribution**, which has a peak to the right of the distribution and data values that taper off to the left. This distribution has a single peak and is also unimodal. For a histogram that is skewed to the left, the mean is located to the left on the distribution and is the smallest value of the measures of central tendency. The mean has the smallest value because it is strongly affected by the outliers on the left tail that pull the mean to the left. The median is located between the mode and the mean.

e) Figure e has no shape that can be defined. The only defining characteristic about this distribution is that it has 2 peaks of the same height. This means that the distribution is bimodal.

While there are similarities between a bar graph and a histogram, such as each bar being the same width, a histogram has no spaces between the bars. The quantitative data is grouped according to a determined bin size, or interval. The bin size refers to the width of each bar, and the data is placed in the appropriate bin.

The **bins**, or groups of data, are plotted on the $x$-axis, and the frequencies of the bins are plotted on the $y$-axis. A grouped **frequency distribution** is constructed for the numerical data, and this table is used to create the histogram. In most cases, the grouped frequency distribution is designed so there are no breaks in the intervals. The last value of one bin is actually the first value counted in the next bin. This means that if you had groups of data with a bin size of 10, the bins would be represented by the notation [0-10), [10-20), [20-30), etc. Each bin appears to contain 11 values, which is 1 more than the desired bin size of 10. Therefore, the last digit of each bin is counted as the first digit of the following bin.
The first bin includes the values 0 through 9, and the next bin includes the values 9 through 19. This makes the bins the proper size. Bin sizes are written in this manner to simplify the process of grouping the data. The first bin can begin with the smallest number of the data set and end with the value determined by adding the bin width to this value, or the bin can begin with a reasonable value that is smaller than the smallest data value.

Constructing a Frequency Distribution Table

1. Construct a frequency distribution table with a bin size of 10 for the following data, which represents the ages of 30 lottery winners:

   38  41  29  33  40  74  66  45  60  55
   25  52  54  61  46  51  59  57  66  62
   32  47  65  50  39  22  35  72  77  49

   **Step 1:** Determine the range of the data by subtracting the smallest value from the largest value.

   \[
   \text{Range: } 77 - 22 = 55
   \]

   **Step 2:** Divide the range by the bin size to ensure that you have at least 5 groups of data. A histogram should have from 5 to 10 bins to make it meaningful: \( \frac{55}{10} = 5.5 \approx 6 \). Since you cannot have 0.5 of a bin, the result indicates that you will have at least 6 bins.

   **Step 3:** Construct the table.

   **Table 10.15:**

<table>
<thead>
<tr>
<th>Bin</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>[20 – 30)</td>
<td>3</td>
</tr>
<tr>
<td>[30 – 40)</td>
<td>5</td>
</tr>
<tr>
<td>[40 – 50)</td>
<td>6</td>
</tr>
<tr>
<td>[50 – 60)</td>
<td>8</td>
</tr>
<tr>
<td>[60 – 70)</td>
<td>5</td>
</tr>
<tr>
<td>[70 – 80)</td>
<td>3</td>
</tr>
</tbody>
</table>
Step 4: Determine the sum of the frequency column to ensure that all the data has been grouped.

\[3 + 5 + 6 + 8 + 5 + 3 = 30\]

When data is grouped in a frequency distribution table, the actual data values are lost. The table indicates how many values are in each group, but it doesn’t show the actual values.

There are many different ways to create a distribution table and many different distribution tables that can be created. However, for the purpose of constructing a histogram, the method shown works very well, and it is not difficult to complete.

2. The numbers of years of service for 75 teachers in a small town are listed below:

1, 6, 11, 26, 21, 18, 2, 5, 27, 33, 7, 15, 22, 30, 8
31, 5, 25, 20, 19, 4, 9, 19, 34, 3, 16, 31, 10, 4
2, 31, 26, 19, 3, 12, 14, 28, 32, 1, 17, 24, 34, 16, 1,
18, 29, 10, 12, 30, 13, 7, 8, 27, 3, 11, 26, 33, 29, 20
7, 21, 11, 19, 35, 16, 5, 2, 19, 24, 13, 14, 28, 10, 31

Using the above data, construct a frequency distribution table with a bin size of 5.

Range: 35 – 1 = 34
\[\frac{34}{5} = 6.8 \approx 7\]

You will have 7 bins.

When the number of data values is very large, another column is often inserted in the distribution table. This column is a tally column, and it is used to account for the number of values within a bin. A tally column facilitates the creation of the distribution table and usually allows the task to be completed more quickly. For each value that is in a bin, draw a stroke in the Tally column. To make counting the strokes easier, draw 4 strokes and cross them out with the fifth stroke. This process bundles the strokes in groups of 5, and the frequency can be readily determined.

**Table 10.16:**

<table>
<thead>
<tr>
<th>Bin</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0 – 5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[5 – 10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[10 – 15)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[15 – 20)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[20 – 25)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[25 – 30)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[30 – 35)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[11 + 9 + 12 + 14 + 7 + 10 + 12 = 75\]

Now that you have constructed the frequency table, the grouped data can be used to draw a histogram. Like a bar
graph, a histogram requires a title and properly labeled x- and y-axes.

**Constructing a Histogram**

Use the data from the first example that displays the ages of the lottery winners to construct a histogram. The data is shown again below. What percentage of the winners were 50 years of age or older?

<table>
<thead>
<tr>
<th>Bin</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>[20 – 30)</td>
<td>3</td>
</tr>
<tr>
<td>[30 – 40)</td>
<td>5</td>
</tr>
<tr>
<td>[40 – 50)</td>
<td>6</td>
</tr>
<tr>
<td>[50 – 60)</td>
<td>8</td>
</tr>
<tr>
<td>[60 – 70)</td>
<td>5</td>
</tr>
<tr>
<td>[70 – 80)</td>
<td>3</td>
</tr>
</tbody>
</table>

Use the data as it is represented in the distribution table to construct the histogram.

From looking at the tops of the bars, you can see how many winners were in each category, and by adding these numbers, you can determine the total number of winners. You can also determine how many winners were within a specific category. For example, you can see that 8 winners were 60 years of age or older. The graph can also be used to determine percentages. For example, it can answer the question, “What percentage of the winners were 50 years of age or older?” as follows:

\[
\frac{16}{30} = 0.533 \quad (0.533)(100\%) \approx 5.3\%.
\]
Examples

Example 1

Use the data and the distribution table that represent the ages of teachers from Example B to construct a histogram to display the data. The distribution table is shown again below:

<table>
<thead>
<tr>
<th>Bin</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0 – 5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[5 – 10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[10 – 15)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[15 – 20)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[20 – 25)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[25 – 30)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[30 – 35)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
</tr>
</tbody>
</table>

![Histogram of Teachers Teaching Time](https://www.ck12.org/flx/render/embeddedobject/137061)
Example 2

\[11 + 9 + 12 + 14 + 7 + 10 + 12 = 75\]
In this small town, 75 teachers are teaching.
How many teachers teach in this small town?

Example 3

11 teachers have taught for less than 5 years.
How many teachers have worked for less than 5 years?

Example 4

If teachers are able to retire when they have taught for 30 years or more, how many are eligible to retire?
12 teachers are eligible to retire.

Example 4

What percentage of the teachers still have to teach for 10 years or fewer before they are eligible to retire?
\[\frac{17}{75} = 0.2266 \quad (0.2266)(100\%) \approx 23\%\]
Approximately 23\% of the teachers must teach for 10 years or fewer before they are eligible to retire.

Example 5

Do you think that the majority of the teachers are young or old? Justify your answer.
Answers will vary, but one possible answer is that the majority of the teachers are young, because 46 have taught for less than 20 years.

Review

1. What name is given to a distribution that has 2 peaks of the same height?
   a. uniform
   b. unimodal
   c. bimodal
   d. discrete

The following histogram shows data collected during a recent fishing derby. The number of fish caught is being compared to the size of the fish caught. Use the histogram to answer the following questions:
2. How many fish were caught?
3. How many fish caught were over 35 cm in length?
4. How many fish caught were between 20 cm and 29 cm in length?
5. Why is there a blank space between 38 cm and 41 cm on the histogram?

The following histogram displays the heights of students in a classroom. Use the information represented in the histogram to answer the following questions:

6. How many students are in the class?
7. How many students are over 60 inches in height?
8. How many students have a height between 54 in and 62 in?
9. Is the distribution unimodal or bimodal? How do you know?
10. The following data represents the results of a test taken by a group of students:

   95  56  70  83  59  66  88  52  50  77  69  80
   54  75  68  78  51  64  55  67  74  57  73  53

Construct a frequency distribution table using a bin size of 10 and display the results in a properly labeled histogram.

**Review (Answers)**

To view the Review answers, open this PDF file and look for section 7.8.
10.11 Applications of Histograms

Here you’ll learn how to use the Texas Instruments calculator to construct histograms.

Applications of Histograms

Technology can also be used to plot a histogram. The TI-83 can be used to create a histogram by using STAT and STAT PLOT on the calculator. The TRACE feature can then be used to find the number of values for each bin, and then a frequency distribution table can be constructed from these values. When constructing a histogram using a TI-83 calculator, it’s important that your settings are correct on the WINDOW screen. The setting that determines the bin size is the Xscl setting, so you’ll want to make sure that the number for Xscl is correct before creating your histogram.

Real-World Application: Piglets

Scientists have invented a new dietary supplement that is supposed to increase the weight of a piglet within its first 3 months of growth. Farmer John fed this supplement to his stock of piglets, and at the end of 3 months, he recorded the weights of 50 randomly selected piglets.

The following table is the recorded weights (in pounds) of the 50 selected piglets:

<table>
<thead>
<tr>
<th>120</th>
<th>111</th>
<th>65</th>
<th>110</th>
<th>114</th>
<th>72</th>
<th>116</th>
<th>105</th>
<th>119</th>
<th>114</th>
</tr>
</thead>
<tbody>
<tr>
<td>93</td>
<td>113</td>
<td>99</td>
<td>118</td>
<td>108</td>
<td>97</td>
<td>107</td>
<td>95</td>
<td>113</td>
<td>75</td>
</tr>
<tr>
<td>84</td>
<td>120</td>
<td>102</td>
<td>104</td>
<td>84</td>
<td>97</td>
<td>121</td>
<td>69</td>
<td>100</td>
<td>101</td>
</tr>
<tr>
<td>107</td>
<td>118</td>
<td>77</td>
<td>105</td>
<td>109</td>
<td>78</td>
<td>89</td>
<td>68</td>
<td>74</td>
<td>103</td>
</tr>
<tr>
<td>87</td>
<td>67</td>
<td>79</td>
<td>90</td>
<td>109</td>
<td>94</td>
<td>106</td>
<td>96</td>
<td>92</td>
<td>88</td>
</tr>
</tbody>
</table>

Using the above data set and your calculator, construct a histogram to represent the data. Use a bin size of 10.

The histogram can be created as follows:
The TRACE Feature

Use the TRACE feature to get information about the data in each bar of the histogram that you created in the previous examples.

First, press

```
TRACE
```

The TRACE feature tells you that in the first bin, which is [60-70), there are 4 values.

The TRACE feature tells you that in the second bin, which is [70-80), there are 6 values.

To advance to the next bin, or bar, of the histogram, use the cursor and move to the right. The numbers of values for the subsequent bins are 5, 9, 13, 10, and 3, respectively. The information obtained by using the TRACE feature will enable you to create a frequency table and to draw the histogram on paper.
Displaying Data in a Histogram and Constructing a Frequency Distribution Table

The following data represents the results of a test taken by a group of students:

95  56  70  83  59  66  88  52  50  77  69  80
54  75  68  78  51  64  55  67  74  57  73  53

Display the results in a histogram using technology and construct a frequency distribution table using a bin size of 10.

The histogram and the numbers of values for the bins are as shown below:

Now the frequency distribution table can be constructed as follows:

<table>
<thead>
<tr>
<th>Bin</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>[40 – 50)</td>
<td>0</td>
</tr>
<tr>
<td>[50 – 60)</td>
<td>9</td>
</tr>
<tr>
<td>[60 – 70)</td>
<td>5</td>
</tr>
<tr>
<td>[70 – 80)</td>
<td>6</td>
</tr>
<tr>
<td>[80 – 90)</td>
<td>3</td>
</tr>
<tr>
<td>[90 – 100)</td>
<td>1</td>
</tr>
</tbody>
</table>
Example

Example 1

The following table is the number of minutes spent practicing per day by a professional violinist for each of the last 80 days:

<table>
<thead>
<tr>
<th>86</th>
<th>67</th>
<th>105</th>
<th>122</th>
<th>98</th>
<th>83</th>
<th>84</th>
<th>101</th>
<th>100</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>93</td>
<td>99</td>
<td>78</td>
<td>101</td>
<td>92</td>
<td>89</td>
<td>95</td>
<td>102</td>
<td>110</td>
<td>100</td>
</tr>
<tr>
<td>84</td>
<td>77</td>
<td>82</td>
<td>107</td>
<td>88</td>
<td>93</td>
<td>94</td>
<td>103</td>
<td>127</td>
<td>111</td>
</tr>
<tr>
<td>120</td>
<td>100</td>
<td>105</td>
<td>95</td>
<td>99</td>
<td>91</td>
<td>103</td>
<td>79</td>
<td>89</td>
<td>84</td>
</tr>
<tr>
<td>112</td>
<td>96</td>
<td>107</td>
<td>106</td>
<td>85</td>
<td>92</td>
<td>105</td>
<td>91</td>
<td>90</td>
<td>104</td>
</tr>
<tr>
<td>89</td>
<td>114</td>
<td>106</td>
<td>99</td>
<td>101</td>
<td>81</td>
<td>92</td>
<td>88</td>
<td>100</td>
<td>97</td>
</tr>
<tr>
<td>78</td>
<td>103</td>
<td>86</td>
<td>107</td>
<td>101</td>
<td>98</td>
<td>77</td>
<td>69</td>
<td>89</td>
<td>90</td>
</tr>
<tr>
<td>98</td>
<td>102</td>
<td>94</td>
<td>103</td>
<td>80</td>
<td>99</td>
<td>109</td>
<td>101</td>
<td>93</td>
<td>86</td>
</tr>
</tbody>
</table>

Display the results in a histogram using technology and construct a frequency distribution table using a bin size of 10.

First, press

\[ \text{STAT} \]

, choose Edit on the EDIT menu, and enter the data values into L1. Since there are so many data values, it’s a good idea to then use 1-Var Stats to find the minimum and the maximum data values so that the correct settings for the histogram can be entered in the WINDOW screen. To access 1-Var Stats, after returning to the main screen, press

\[ \text{STAT} \]

and choose 1-Var Stats on the CALC menu. After pressing

\[ \text{ENTER} \]

to choose 1-Var Stats, press

\[ \text{ENTER} \]

again, and then use the down arrow to find the minimum and maximum values. You should see the following:
Now that we know that the minimum value is 67 and the maximum value is 127, we can enter the settings for the histogram in the WINDOW screen. To do this, press

\[ \text{WINDOW} \]

and enter what is shown below:

\[
\begin{align*}
\text{WINDOW} \\
\text{Xmin}=60 \\
\text{Xmax}=130 \\
\text{Xscl}=10 \\
\text{Ymin}=0 \\
\text{Ymax}=30 \\
\text{Yscl}=3 \\
\text{Xres}=1
\end{align*}
\]

We are now ready to create our histogram. To do so, press

\[ \text{2ND} \quad \text{Y=} \quad \text{ENTER} \]

, and make sure you have the following settings:

\[
\begin{align*}
\text{Plot1} & : \text{On} \\
\text{Type} & : \text{L} \\
\text{Xlist} & : \text{L1} \\
\text{Freq} & : \text{1}
\end{align*}
\]

Finally, press

\[ \text{GRAPH} \]

, and you should see the histogram shown below:
At this point, you can use the TRACE feature to find the number of values for each bin. Just press

\[ \text{TRACE} \]

and use the right arrow to move across the screen.

The numbers for the bins should be 2, 5, 17, 23, 25, 4, and 4, respectively. Notice that \( 2 + 5 + 17 + 23 + 25 + 4 + 4 = 80 \), which is the number of days for which the data exists. Since we now have the frequency for each bin, the frequency distribution table can be created as follows:

**Table 10.20:**

<table>
<thead>
<tr>
<th>Bin</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>[60 – 70)</td>
<td>2</td>
</tr>
<tr>
<td>[70 – 80)</td>
<td>5</td>
</tr>
<tr>
<td>[80 – 90)</td>
<td>17</td>
</tr>
<tr>
<td>[90 – 100)</td>
<td>23</td>
</tr>
<tr>
<td>[100 – 110)</td>
<td>25</td>
</tr>
<tr>
<td>[110 – 120)</td>
<td>4</td>
</tr>
<tr>
<td>[120 – 130)</td>
<td>4</td>
</tr>
</tbody>
</table>

**Review**

Use the histogram shown below to answer the following questions:
10.11. Applications of Histograms

1. What is the bin size for the histogram?
2. Which bin(s) have the highest frequency?
3. Which bin(s) have the lowest frequency?
4. What is the total number of data values represented by the histogram?
5. What percentage of the data values are in the bin [60-70)?

Use the histogram shown below to answer the following questions:

6. What is the bin size for the histogram?
7. Which bin(s) have the highest frequency?
8. Which bin(s) have the lowest frequency?
9. What is the total number of data values represented by the histogram?
10. What percentage of the data values are in the bin [15-20)?

Review (Answers)

To view the Review answers, open this PDF file and look for section 7.9.
Here you’ll learn how to answer questions about data by looking for trends in the data and by viewing the data in tabular form.

Suppose that everyday you take the same number of vitamins so that the number of vitamins remaining in the bottle always decreases by the same amount. If you know how many vitamins were in the bottle to begin with, do you think you can determine the number of vitamins in the bottle after a certain number of days? Would it be easier if you looked at the data in a table, with the number of days in one column and the number of vitamins remaining in the bottle in another column?

### Trends in Data

#### Problem-Solving Strategies: Make a Table or Look for a Pattern

This lesson focuses two strategies to solve problems: making a table and looking for a pattern. These are the most common strategies you have used before algebra. Let’s review the four-step problem-solving plan from the previous Concept:

- **Step 1:** Understand the problem.
- **Step 2:** Devise a plan - Translate. Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart, or construct a table as a start to begin your problem-solving plan.
- **Step 3:** Carry out the plan - Solve.
- **Step 4:** Check and Interpret: Check to see if you used all your information. Then look to see if the answer makes sense.

#### Solving a Problem by Using a Table

When a problem has data that needs to be organized, a table is a highly effective problem-solving strategy. A table is also helpful when the problem asks you to record a large amount of information. Patterns and numerical relationships are easier to see when data are organized in a table.

**Let’s use a table to solve the following problem:**

Josie takes up jogging. In the first week she jogs for 10 minutes per day, and in the second week she jogs for 12 minutes per day. Each week, she wants to increase her daily jogging time by 2 minutes. If she jogs six days per week each week, what will be her total jogging time in the sixth week?

Organize the information in a table

<table>
<thead>
<tr>
<th>Week 1</th>
<th>Week 2</th>
<th>Week 3</th>
<th>Week 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 minutes</td>
<td>12 minutes</td>
<td>14 minutes</td>
<td>16 minutes</td>
</tr>
<tr>
<td>60 min/week</td>
<td>72 min/week</td>
<td>84 min/week</td>
<td>96 min/week</td>
</tr>
</tbody>
</table>

We can see the pattern that the number of minutes is increasing by 12 each week. Continuing this pattern, Josie will run 120 minutes in the sixth week.

Don’t forget to check the solution! The pattern starts at 60 and adds 12 each week after the first week. The equation
to represent this situation is \( t = 60 + 12(w - 1) \). By substituting 6 for the variable of \( w \), the equation becomes 
\[
 t = 60 + 12(6 - 1) = 60 + 60 = 120.
\]

**Solving a Problem by Looking for a Pattern**

Some situations have a readily apparent pattern, which means that the pattern is easy to see. In this case, you may not need to organize the information into a table. Instead, you can use the pattern to arrive at your solution.

**Let’s look for a pattern to solve the following problem:**

You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 layers?

One layer: It is simple to see that a triangle with one layer has only one ball.

Two layers: For a triangle with two layers we add the balls from the top layer to the balls of the bottom layer. It is useful to make a sketch of the different layers in the triangle.

Three layers: We add the balls from the top triangle to the balls from the bottom layer.

We can fill the first three rows of the table.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 6 + 4 = 10 \\
\end{array}
\]

To find the number of tennis balls in 8 layers, continue the pattern.

\[
\begin{array}{cccc}
5 & 6 & 7 & 8 \\
10 + 5 = 15 & 15 + 6 = 21 & 21 + 7 = 28 & 28 + 8 = 36 \\
\end{array}
\]
There will be 36 tennis balls in the 8 layers.

Check: Each layer of the triangle has one more ball than the previous one. In a triangle with 8 layers, each layer has the same number of balls as its position. When we add these we get:

\[ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36 \text{ balls} \]

The answer checks out.

Now, let’s compare the two methods, making a table and using a pattern, in solving the following problem:

Andrew cashes a $180 check and wants the money in $10 and $20 bills. The bank teller gives him 12 bills. How many of each kind of bill does he receive?

Method 1: Making a Table

<table>
<thead>
<tr>
<th>Tens</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Twenties</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The combination that has a sum of 12 is six $10 bills and six $20 bills.

Method 2: Using a Pattern

The pattern is that for every pair of $10 bills, the number of $20 bills is reduced by one. Begin with the most number of $20 bills. For every $20 bill lost, add two $10 bills.

\[ 6(\$10) + 6(\$20) = \$180 \]

Six $10 bills and six $20 bills = \$60 + \$120 = \$180.

MEDIA

Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/56121

MEDIA

Click image to the left or use the URL below.
URL: https://www.ck12.org/flx/render/embeddedobject/179416
Examples

Example 1

Earlier, you were told that you take the same number of vitamins a day so that the number of vitamins remaining in
the bottle always decreases by the same amount. If you know how many vitamins were in the bottle to begin with,
can you determine the number of vitamins in the bottle after a certain number of days? Would it be easier to look at
the data in a table?

You can determine the number of vitamins in the bottle after a certain amount of days because the amount decreases
by the same number each day. There would be a clear pattern. A table would make the it easier to find a pattern but
it is not necessary.

Example 2

Students are going to march in a homecoming parade. There will be one kindergartener, two first-graders, three
second-graders, and so on through 12th grade. How many students will be walking in the homecoming parade?


Make a table:

\[
\begin{array}{cccccccccccc}
K & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\end{array}
\]

The solution is the sum of all the numbers, 91. There will be 91 students walking in the homecoming parade.

Look for a pattern.

The pattern is that the number of students is one more than their grade level. Therefore, the solution is the sum of
the numbers from 1 (kindergarten) through 13 (12th grade). The solution is 91.

Review

1. In the first problem in this concept, you were told that Josie jogs for 10 minutes each day in the first week, 12
   minutes each day in the second week, and after that increases her daily jogging time by 2 minutes each week.
   Josie jogs six days per week each week. What will Josie’s total jogging time be in the in the beginning in the
eighth week?
2. Britt has $2.25 in nickels and dimes. If she has 40 coins in total how many of each coin does she have?
3. A pattern of squares is placed together as shown. How many squares are in the 12th diagram?

4. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts with
   24 cups the first week, cuts down to 21 cups the second week, and drops to 18 cups the third week, how many
   weeks will it take him to reach his goal?
5. Taylor checked out a book from the library and it is now 5 days late. The late fee is 10 cents per day. How
   much is the fine?
6. How many hours will a car traveling at 75 miles per hour take to catch up to a car traveling at 55 miles per hour if the slower car starts two hours before the faster car?
7. Grace starts biking at 12 miles per hour. One hour later, Dan starts biking at 15 miles per hour, following the same route. How long would it take him to catch up with Grace?
8. Lemuel wants to enclose a rectangular plot of land with a fence. He has 24 feet of fencing. What is the largest possible area that he could enclose with the fence?

Mixed Review

9. Determine if the relation is a function: \{ (2,6), (-9,0), (7,7), (3,5), (5,3) \}.
10. Roy works construction during the summer and earns $78 per job. Create a table relating the number of jobs he could work, \( j \), and the total amount of money he can earn, \( m \).
11. Graph the following order pairs: (4,4); (-5,6), (-1,-1), (-7,-9), (2,-5).
12. Evaluate the following expression: \(-4(4z - x + 5)\); use \( x = -10 \) and \( z = -8 \).
13. The area of a circle is given by the formula \( A = \pi r^2 \). Determine the area of a circle with radius 6 mm.
14. Louie bought 9 packs of gum at $1.19 each. How much money did he spend?
15. Write the following without the multiplication symbol: \( 16 \times \frac{1}{8}c. \)

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.16.
Here you’ll learn how to collect accurate data with surveys and samples.

What if you wanted to know the percentage of the people in your town who eat breakfast at least 5 times per week? Your town is too big to ask each person individually, so what would you do? How could you ensure that the information you find out is accurate?

**Surveys and Samples**

One of the most important applications of statistics is collecting information. Statistical studies are done for many purposes:

- To find out more about animal behaviors;
- To determine which presidential candidate is favored;
- To figure out what type of chip product is most popular;
- To determine the gas consumption of cars.

In most cases, it is not possible to survey everyone in a given population. So a **sample** is taken. It is essential that the sample is a **representative sample** of the population being studied. For example, if we are trying to determine the effect of a drug on teenage girls, it would make no sense to include males in our sample population, nor would it make sense to include women that are not teenagers.

**Sampling Methods**

The two types of sampling methods we will cover are:

- Random Sampling
- Stratified Sampling

**Random sampling** is a method in which people are chosen “out of the blue.” In a true random sample, everyone in the population must have the same chance of being chosen. It is important that each person in the population has a chance of being picked.

**Stratified sampling** is a method actively seeking to poll people from many different backgrounds. The population is first divided into different categories (or **strata**) and the number of members in each category is determined. In order to lessen the chance of a biased result, the sample size must be large enough. The larger the sample size is, the more precise the estimate is. However, the larger the sample size, the more expensive and time-consuming the statistical study becomes.

**For the following problem, let’s choose the best sampling method:**

For a class assignment you have been asked to find if students in your school are planning to attend university after graduating high school. Students can respond with “yes,” “no,” or “undecided.” How will you choose those you wish to interview if you want your results to be reliable?

The stratified sampling method would be the best option. By randomly picking a certain number of students in each grade, you will get the most accurate results.
Biased Samples

If the sample ends up with one or more sub-groups that are either over-represented or under-represented, then we say the sample is biased. We would not expect the results of a biased sample to represent the entire population, so it is important to avoid selecting a biased sample.

Some samples may deliberately seek a biased sample in order to obtain a particular viewpoint. For example, if a group of students were trying to petition the school to allow eating candy in the classroom, they might only survey students immediately before lunchtime when students are hungry. The practice of polling only those who you believe will support your cause is sometimes referred to as cherry picking.

Many surveys may have a non-response bias. In this case, a survey that is simply handed out gains few responses when compared to the number of surveys given out. People who are either too busy or simply not interested will be excluded from the results. Non-response bias may be reduced by conducting face-to-face interviews.

Self-selected respondents who tend to have stronger opinions on subjects than others and are more motivated to respond may also cause bias. For this reason phone-in and online polls also tend to be poor representations of the overall population. Even though it appears that both sides are responding, the poll may disproportionately represent extreme viewpoints from both sides, while ignoring more moderate opinions that may, in fact, be the majority view. Self-selected polls are generally regarded as unscientific.

Now, let’s take a look at determining bias:

Determine whether the following survey is biased. Explain your reasoning.

“Asking people shopping at a farmer’s market if they think locally grown fruit and vegetables are healthier than supermarket fruits and vegetables”

This would be a biased sample because people shopping at a farmer’s market are generally interested in buying fresher fruits and vegetables than a regular supermarket provides. The study can be improved by interviewing an equal number of people coming out of a supermarket, or by interviewing people in a more neutral environment such as the post office.

Biased Questions

Although your sample may be a good representation of the population, the way questions are worded in the survey can still provoke a biased result. There are several ways to identify biased questions.

1. They may use polarizing language, words, and phrases that people associate with emotions.
   - How much of your time do you waste on TV every week?
2. They may refer to a majority or to a supposed authority.
   - Would you agree with the American Heart and Lung Association that smoking is bad for your health?
3. They may be phrased so as to suggest the person asking the question already knows the answer to be true, or to be false.
   - You wouldn’t want criminals free to roam the streets, would you?
4. They may be phrased in an ambiguous way (often with double negatives), which may confuse people.
   - Do you disagree with people who oppose the ban on smoking in public places?

### Design and Conduct a Survey

The method in which you design and conduct the survey is crucial to its accuracy. Surveys are a set of questions in which the sample answers. The data is compiled to form results, or findings. When designing a survey, be aware of the following recommendations.

1. Determine the goal of your survey. What question do you want to answer?
2. Identify the sample population. Who will you interview?
3. Choose an interviewing method, face-to-face interview, phone interview, or self-administered paper survey or internet survey.
4. Conduct the interview and collect the information.
5. Analyze the results by making graphs and drawing conclusions.

Surveys can be conducted in several ways.

### Face-to-face interviews

- Fewer misunderstood questions
- High response rate
- Additional information can be collected from respondents
  - Time-consuming
  - Expensive
  - Can be biased based upon the attitude or appearance of the surveyor

### Self-administered surveys

- Respondent can complete on his or her free time
- Less expensive than face-to-face interviews
- Anonymity causes more honest results
  - Lower response rate

### Let’s take a look at the steps involved in constructing a survey.

Martha wants to construct a survey that shows which sports students at her school like to play the most.

1. List the goal of the survey.

   The goal of the survey is to find the answer to the question: “Which sports do students at Martha’s school like to play the most?”
2. What population sample should she interview?

A sample of the population would include a random sample of the student population in Martha’s school. A good strategy would be to randomly select students (using dice or a random number generator) as they walk into an all-school assembly.

3. How should she administer the survey?

Face-to-face interviews are a good choice in this case since the survey consists of only one question, which can be quickly answered and recorded.

4. Create a data collection sheet that she can use to record her results.

In order to collect the data to this simple survey, Martha can design a data collection sheet such as the one below:

**Display, Analyze, and Interpret Survey Data**

We have shown you several ways to display data. These graphs are also useful when displaying survey results. Survey data can be displayed as:

- A bar graph
- A histogram
- A pie chart
- A tally sheet
- A box-and-whisker plot
- A stem-and-leaf plot

The method in which you choose to display your data will depend upon your survey results and to whom you plan to present the data.

---

**Examples**

**Example 1**

Earlier, we asked about the percentage of people in your town who eat breakfast at least 5 times per week. Your town is too big to ask each person individually, so what would you do? How could you ensure that the information you find out is accurate?

Because your town is too big to ask each person individually, you could conduct a random sample. In order to ensure that the information you collect is accurate, you can use a non-biased survey question.
Example 2

Raoul wants to construct a survey that shows how many hours per week the average student at his school works.

1. List the goal of the survey.
2. What population sample will he interview?
3. How would he administer the survey?

The goal of the survey is to determine the number of hours the average student at Raoul’s school works.
The population to be surveyed is the student body.

Care should be taken to randomly select students, so that variety of student life is represented. Surveying the football team only would not be a good representation of the whole student body. Students can be randomly surveyed, using dice or a random number generator, in a setting where all, or most, students will be, such as an assembly.

Review

1. Explain the most common types of sampling methods. If you needed to survey a city about a new road project, which sampling method would you choose? Explain.
2. What is a biased survey? How can bias be avoided?
3. How are surveys conducted, according to this text? List one advantage and one disadvantage of each. List one additional method that can be used to conduct surveys.
4. What are some keys to recognizing biased questions? What could you do if you were presented with a biased question?
5. For a class assignment, you have been asked to find out how students get to school. Do they take public transportation, drive themselves, have their parents drive them, use a carpool, or walk/bike? You decide to interview a sample of students. How will you choose those you wish to interview if you want your results to be reliable?
6. Comment on the way the following samples have been chosen. For the unsatisfactory cases, suggest a way to improve the sample choice.
   a. You want to find whether wealthier people have more nutritious diets by interviewing people coming out of a five-star restaurant.
   b. You want to find if a pedestrian crossing is needed at a certain intersection by interviewing people walking by that intersection.
   c. You want to find out if women talk more than men by interviewing an equal number of men and women.
   d. You want to find whether students in your school get too much homework by interviewing a stratified sample of students from each grade level.
   e. You want to find out whether there should be more public busses running during rush hour by interviewing people getting off the bus.
   f. You want to find out whether children should be allowed to listen to music while doing their homework by interviewing a stratified sample of male and female students in your school.

7. Raoul found that 30% of the students at his school are in 9th grade, 26% of the students are in the 10th grade, 24% of the students are in 11th grade, and 20% of the students are in the 12th grade. He surveyed a total of 60 students using these proportions as a guide for the number of students he interviewed from each grade. Raoul recorded the following data.
   a. Construct a stem-and-leaf plot of the collected data.
   b. Construct a frequency table with a bin size of 5.
   c. Draw a histogram of the data.
   d. Find the five-number summary of the data and draw a box-and-whisker plot.
### Table 10.22:

<table>
<thead>
<tr>
<th>Grade Level</th>
<th>Record Worked</th>
<th>Number of Hours</th>
<th>Total Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>9th grade</td>
<td>0, 5, 4, 0, 0, 10, 5, 6, 0, 0, 2, 4, 0, 8, 0, 5, 7, 0</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>10th grade</td>
<td>6, 10, 12, 0, 10, 15, 0, 0, 8, 5, 0, 7, 10, 12, 0, 0</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>11th grade</td>
<td>0, 12, 15, 18, 10, 0, 0, 20, 8, 15, 10, 15, 0, 5</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>12th grade</td>
<td>22, 15, 12, 15, 10, 0, 18, 20, 10, 0, 12, 16</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

8. The following pie chart displays data from a survey asking students the type of sports they enjoyed playing most. Make five conclusions regarding the survey results.

9. Melissa conducted a survey to answer the question: “What sport do high school students like to watch on TV the most?” She collected the following information on her data collection sheet.
   a. Make a pie chart of the results showing the percentage of people in each category.
   b. Make a bar-graph of the results.

### Table 10.23:

<table>
<thead>
<tr>
<th>Sport</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseball</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>⌈ guitarist ⌋</td>
</tr>
<tr>
<td>Basketball</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>⌈ guitarist ⌋</td>
</tr>
</tbody>
</table>
10.13. Surveys and Samples

**TABLE 10.23:** (continued)

<table>
<thead>
<tr>
<th>Sport</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>Football</td>
<td>24</td>
</tr>
<tr>
<td>Soccer</td>
<td>18</td>
</tr>
<tr>
<td>Gymnastics</td>
<td>19</td>
</tr>
<tr>
<td>Figure Skating</td>
<td>8</td>
</tr>
<tr>
<td>Hockey</td>
<td>18</td>
</tr>
</tbody>
</table>

Total 147

10. Samuel conducted a survey to answer the following question: “What is the favorite kind of pie of the people living in my town?” By standing in front of his grocery store, he collected the following information on his data collection sheet: #Make a pie chart of the results showing the percentage of people in each category.

a. Make a bar graph of the results.

**TABLE 10.24:**

<table>
<thead>
<tr>
<th>Type of Pie</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apple</td>
<td>37</td>
</tr>
<tr>
<td>Pumpkin</td>
<td>13</td>
</tr>
<tr>
<td>Lemon Meringue</td>
<td>7</td>
</tr>
<tr>
<td>Chocolate Mousse</td>
<td>23</td>
</tr>
</tbody>
</table>
TABLE 10.24: (continued)

<table>
<thead>
<tr>
<th>Type of Pie</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cherry</td>
<td>4</td>
</tr>
<tr>
<td>Chicken Pot Pie</td>
<td>31</td>
</tr>
<tr>
<td>Other</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Total 122</td>
</tr>
</tbody>
</table>

11. Myra conducted a survey of people at her school to see “In which month does a person’s birthday fall?” She collected the following information in her data collection sheet:
   a. Make a pie chart of the results showing the percentage of people whose birthday falls in each month.
   b. Make a bar graph of the results.

TABLE 10.25:

<table>
<thead>
<tr>
<th>Month</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>16</td>
</tr>
<tr>
<td>February</td>
<td>13</td>
</tr>
<tr>
<td>March</td>
<td>12</td>
</tr>
<tr>
<td>April</td>
<td>11</td>
</tr>
<tr>
<td>May</td>
<td>13</td>
</tr>
</tbody>
</table>
### Table 10.25: (continued)

<table>
<thead>
<tr>
<th>Month</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>June</td>
<td>12</td>
</tr>
<tr>
<td>July</td>
<td>9</td>
</tr>
<tr>
<td>August</td>
<td>7</td>
</tr>
<tr>
<td>September</td>
<td>9</td>
</tr>
<tr>
<td>October</td>
<td>8</td>
</tr>
<tr>
<td>November</td>
<td>13</td>
</tr>
<tr>
<td>December</td>
<td>13</td>
</tr>
</tbody>
</table>

Total: 136

12. Nam-Ling conducted a survey that answers the question: “Which student would you vote for in your school’s elections?” She collected the following information:

a. Make a pie chart of the results showing the percentage of people planning to vote for each candidate.
b. Make a bar graph of the results.

### Table 10.26:

<table>
<thead>
<tr>
<th>Candidate</th>
<th>9&lt;sup&gt;th&lt;/sup&gt; graders</th>
<th>10&lt;sup&gt;th&lt;/sup&gt; graders</th>
<th>11&lt;sup&gt;th&lt;/sup&gt; graders</th>
<th>12&lt;sup&gt;th&lt;/sup&gt; graders</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Susan Cho</td>
<td>19</td>
<td>57</td>
<td>10</td>
<td>21</td>
<td>19</td>
</tr>
</tbody>
</table>

570
TABLE 10.26: (continued)

<table>
<thead>
<tr>
<th>Candidate</th>
<th>9th graders</th>
<th>10th graders</th>
<th>11th graders</th>
<th>12th graders</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Margarita Martinez</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steve Coogan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solomon Duning</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Juan Rios</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>36</td>
<td>30</td>
<td>30</td>
<td>24</td>
<td>120</td>
</tr>
</tbody>
</table>

13. Graham conducted a survey to find how many hours of TV teenagers watch each week in the United States. He collaborated with three friends who lived in different parts of the U.S. and found the following information:
   a. Make a stem-and-leaf plot of the data.
   b. Decide on an appropriate bin size and construct a frequency table.
   c. Make a histogram of the results.
   d. Find the five-number summary of the data and construct a box-and-whisker plot.

TABLE 10.27:

<table>
<thead>
<tr>
<th>Part of the country</th>
<th>Number of hours of TV watched per week</th>
<th>Total number of teens</th>
</tr>
</thead>
<tbody>
<tr>
<td>West Coast</td>
<td>10, 12, 8, 20, 6, 0, 15, 18, 12, 22, 9, 5, 16, 12, 10, 18, 10, 20, 24, 8</td>
<td>20</td>
</tr>
<tr>
<td>Mid West</td>
<td>20, 12, 24, 10, 8, 26, 34, 15, 18, 6, 22, 16, 10, 20, 15, 25, 32, 12, 18, 22</td>
<td>20</td>
</tr>
<tr>
<td>New England</td>
<td>16, 9, 12, 0, 6, 10, 15, 24, 20, 30, 15, 10, 12, 8, 28, 32, 24, 12, 10, 10</td>
<td>20</td>
</tr>
<tr>
<td>South</td>
<td>24, 22, 12, 32, 30, 20, 25, 15, 10, 14, 10, 12, 24, 28, 32, 38, 20, 25, 15, 12</td>
<td>20</td>
</tr>
</tbody>
</table>

14. “What do students in your high school like to spend their money on?”
   a. Which categories would you include on your data collection sheet?
   b. Design the data collection sheet that can be used to collect this information.
   c. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
   d. Make a pie chart of the results showing the percentage of people in each category.
   e. Make a bar graph of the results.
15. “What is the height of students in your class?”
   a. Design the data collection sheet that can be used to collect this information.
   b. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
   c. Make a stem-and-leaf plot of the data.
   d. Decide on an appropriate bin size and construct a frequency table.
   e. Make a histogram of the results.
   f. Find the five-number summary of the data and construct a box-and-whisker plot.

16. “How much allowance money do students in your school get per week?”
   a. Design the data collection sheet that can be used to collect this information.
   b. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
   c. Make a stem-and-leaf plot of the data.
   d. Decide on an appropriate bin size and construct a frequency table.
   e. Make a histogram of the results.
   f. Find the five-number summary of the data and construct a box-and-whisker plot.

17. Are the following statements biased?
   a. You want to find out public opinion on whether teachers get paid a sufficient salary by interviewing the teachers in your school.
   b. You want to find out if your school needs to improve its communications with parents by sending home a survey written in English.

18. “What time do students in your school get up in the morning during the school week?”
   a. Design the data collection sheet that can be used to collect this information.
   b. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
   c. Make a stem-and-leaf plot of the data.
   d. Decide on an appropriate bin size and construct a frequency table.
   e. Make a histogram of the results.
   f. Find the five-number summary of the data and construct a box-and-whisker plot.

Mixed Review

19. Write the equation containing (8, 1) and (4, -6) in point-slope form.
   a. What is the equation for the line perpendicular to this containing (0, 0)?
   b. What is the equation for the line parallel to this containing (4, 0)?

20. Classify $\sqrt[3]{64}$ according to the real number hierarchy.

21. A ferry traveled to its destination, 22 miles across the harbor. On the first voyage, the ferry took 45 minutes. On the return trip, the ferry encountered a head wind and its trip took one hour, ten minutes. Find the speed of the ferry and the speed of the wind.

22. Solve for $a$: $\frac{6a}{a-1} = \frac{7}{a^2+7}$.

23. Simplify $\frac{7}{187} \cdot \frac{98}{14}$.

24. Use long division to simplify: $\frac{3w^3-6w^2-27w+54}{2w^2-4w-30}$.

25. A hot air balloon rises 16 meters every second.
   a. Is this an example of a linear function, a quadratic function, or an exponential function? Explain.
   b. At four seconds the balloon is 68.5 meters from the ground. What was its beginning height?

Review (Answers)

To see the Review answers, open this PDF file and look for section 12.10.
10.14 Levels of Measurement

- Understand the difference between the levels of measurement: nominal, ordinal, interval, and ratio.

Levels of Measurement

Some researchers and social scientists use a more detailed distinction of measurement, called the levels of measurement, when examining the information that is collected for a variable. This widely accepted (though not universally used) theory was first proposed by the American psychologist Stanley Smith Stevens in 1946. According to Stevens’ theory, the four levels of measurement are nominal, ordinal, interval, and ratio.

Each of these four levels refers to the relationship between the values of the variable.

Nominal measurement

A nominal measurement is one in which the values of the variable are names.

Ordinal measurement

An ordinal measurement involves collecting information of which the order is somehow significant. The name of this level is derived from the use of ordinal numbers for ranking (1st, 2nd, 3rd, etc.).

Examples of Nominal and Ordinal Measurements

The names of the different species of Galapagos tortoises are an example of a nominal measurement.

If we measured the different species of tortoise from the largest population to the smallest, this would be an example of ordinal measurement. In ordinal measurement, the distance between two consecutive values does not have meaning. The 1st and 2nd largest tortoise populations by species may differ by a few thousand individuals, while the 7th and 8th may only differ by a few hundred.

Interval measurement

With interval measurement, there is significance to the distance between any two values.

Ratio measurement

A ratio measurement is the estimation of the ratio between a magnitude of a continuous quantity and a unit magnitude of the same kind. A variable measured at this level not only includes the concepts of order and interval, but also adds the idea of ‘nothingness’, or absolute zero.

Examples of Interval and Ratio Measurement

We can use examples of temperature for these.

An example commonly cited for interval measurement is temperature (either degrees Celsius or degrees Fahrenheit). A change of 1 degree is the same if the temperature goes from 0°C to 1°C as it is when the temperature goes from 40°C to 41°C. In addition, there is meaning to the values between the ordinal numbers. That is, a half of a degree has meaning.
10.14. Levels of Measurement

With the temperature scale of the previous example, 0°C is really an arbitrarily chosen number (the temperature at which water freezes) and does not represent the absence of temperature. As a result, the ratio between temperatures is relative, and 40°C, for example, is not twice as hot as 20°C. On the other hand, for the Galapagos tortoises, the idea of a species having a population of 0 individuals is all too real! As a result, the estimates of the populations are measured on a ratio level, and a species with a population of about 3,300 really is approximately three times as large as one with a population near 1,100.

Comparing the Levels of Measurement

Using Stevens’ theory can help make distinctions in the type of data that the numerical/categorical classification could not. Let’s use an example from the previous section to help show how you could collect data at different levels of measurement from the same population.

Determining Levels of Measurement

Assume your school wants to collect data about all the students in the school.

If we collect information about the students’ gender, race, political opinions, or the town or sub-division in which they live, we have a nominal measurement.

If we collect data about the students’ year in school, we are now ordering that data numerically (9th, 10th, 11th, or 12th grade), and thus, we have an ordinal measurement.

If we gather data for students’ SAT math scores, we have an interval measurement. There is no absolute 0, as SAT scores are scaled. The ratio between two scores is also meaningless. A student who scored a 600 did not necessarily do twice as well as a student who scored a 300.

Data collected on a student’s age, height, weight, and grades will be measured on the ratio level, so we have a ratio measurement. In each of these cases, there is an absolute zero that has real meaning. Someone who is 18 years old is twice as old as a 9-year-old.

It is also helpful to think of the levels of measurement as building in complexity, from the most basic (nominal) to the most complex (ratio). Each higher level of measurement includes aspects of those before it. The diagram below is a useful way to visualize the different levels of measurement.
Example

Use the approximate distribution of Giant Galapagos Tortoises in 2004 to answer the following questions.

**Table 10.28:**

<table>
<thead>
<tr>
<th>Island or Volcano</th>
<th>Species</th>
<th>Climate Type</th>
<th>Shell Shape</th>
<th>Estimate of Total Population</th>
<th>Population Density (per km²)</th>
<th>Number of Repatriated*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wolf</td>
<td>becki</td>
<td>semi-arid</td>
<td>intermediate</td>
<td>1139</td>
<td>228</td>
<td>40</td>
</tr>
<tr>
<td>Darwin</td>
<td>microphyes</td>
<td>semi-arid</td>
<td>dome</td>
<td>818</td>
<td>205</td>
<td>0</td>
</tr>
<tr>
<td>Alcedo</td>
<td>vanden-burghii</td>
<td>humid</td>
<td>dome</td>
<td>6,320</td>
<td>799</td>
<td>0</td>
</tr>
<tr>
<td>Sierra Negra</td>
<td>guntheri</td>
<td>humid</td>
<td>flat</td>
<td>694</td>
<td>122</td>
<td>286</td>
</tr>
<tr>
<td>Cerro Azul</td>
<td>vicina</td>
<td>humid</td>
<td>dome</td>
<td>2,574</td>
<td>155</td>
<td>357</td>
</tr>
<tr>
<td>Santa Cruz</td>
<td>nigrita</td>
<td>humid</td>
<td>dome</td>
<td>3,391</td>
<td>730</td>
<td>210</td>
</tr>
<tr>
<td>Española</td>
<td>hoodensis</td>
<td>arid</td>
<td>saddle</td>
<td>869</td>
<td>200</td>
<td>1,293</td>
</tr>
<tr>
<td>San Cristóbal</td>
<td>chathamen-sis</td>
<td>semi-arid</td>
<td>dome</td>
<td>1,824</td>
<td>559</td>
<td>55</td>
</tr>
<tr>
<td>Santiago</td>
<td>darwini</td>
<td>humid</td>
<td>intermediate</td>
<td>1,165</td>
<td>124</td>
<td>498</td>
</tr>
<tr>
<td>Pinzón</td>
<td>ephippium</td>
<td>arid</td>
<td>saddle</td>
<td>532</td>
<td>134</td>
<td>552</td>
</tr>
<tr>
<td>Pinta</td>
<td>abingdoni</td>
<td>arid</td>
<td>saddle</td>
<td>1</td>
<td>Does not apply</td>
<td>0</td>
</tr>
</tbody>
</table>

**Example 1**

What is the highest level of measurement that could be correctly applied to the variable ’Population Density’?

1. Nominal
2. Ordinal
3. Interval
4. Ratio

Population density is quantitative data, which means it will either fall into the nominal or ordinal categories. Now we just have to think about whether it has a true zero. Does a population density of 0 mean that there really is no population density? Yes, that is the correct meaning, so it is a true zero. This means that the highest level of measurement is ratio.

Note: If you are curious about the “does not apply” in the last row of Table 3, read on! There is only one known individual Pinta tortoise, and he lives at the Charles Darwin Research station. He is affectionately known as...
Lonesome George. He is probably well over 100 years old and will most likely signal the end of the species, as attempts to breed have been unsuccessful.

**Review**

For 1-4, identify the level(s) at which each of these measurements has been collected.

1. Lois surveys her classmates about their eating preferences by asking them to rank a list of foods from least favorite to most favorite.
2. Lois collects similar data, but asks each student what her favorite thing to eat is.
3. In math class, Noam collects data on the Celsius temperature of his cup of coffee over a period of several minutes.
4. Noam collects the same data, only this time using degrees Kelvin.

For 5-8, explain whether or not the following statements are true.

5. All ordinal measurements are also nominal.
6. All interval measurements are also ordinal.
7. All ratio measurements are also interval.
8. Steven’s levels of measurement is the one theory of measurement that all researchers agree on.

For 9-11, indicate whether the variable is ordinal or not. If the variable is not ordinal, indicate its variable type.

9. Opinion about a new law (favor or oppose)
10. Letter grade in an English class (A, B, C, etc.)
11. Student rating of teacher on a scale of 1 - 10.

For 12-14, explain whether the quantitative variable is continuous or not:

12. Time it takes for student to get from home to school
13. Number of hours a student studies per night
14. Height (in inches)

15. Give an example of an ordinal variable for which the average would make sense as a numerical summary.
16. Find an example of a study in a magazine, newspaper or website. Determine what variables were measured and for each variable determine its type.
17. How do we summarize, display, and compare data measured at different levels?

**Review (Answers)**

To view the Review answers, open this PDF file and look for section 1.2.
Here you’ll learn how to use a frequency distribution table to calculate the mean or average of a set of ungrouped data. You’ll also learn how to calculate the mean of a set of data using technology.

**Ungrouped Data to Find the Mean**

When a data set is large, a frequency distribution table is often used to display the data in an organized way. A frequency distribution table lists the data values, as well as the number of times each value appears in the data set. A frequency distribution table is easy to both read and interpret and in this concept is used for ungrouped data, or data that is listed.

The numbers in a frequency distribution table do not have to be put in order. To make it easier to enter the values in the table, a tally column is often inserted. Inserting a tally column allows you to account for every value in the data set, without having to continually scan the numbers to find them in the list. A slash (/) is used to represent the presence of a value in the list, and the total number of slashes will be the frequency. If a tally column is inserted, the table will consist of 3 columns, and if no tally column is inserted, the table will consist of 2 columns. The formula that was written to determine the mean, \( \bar{x} = \frac{\sum x_1 + x_2 + x_3 + \ldots + x_n}{n} \), does not show any multiplication of the numbers by their frequencies. However, this can be easily inserted into this formula as shown below:

\[
\bar{x} = \frac{\sum x_1 f_1 + x_2 f_2 + x_3 f_3 + \ldots + x_n f_n}{f_1 + f_2 + f_3 + \ldots + f_n}
\]

Let’s examine this concept with an actual problem and data.

**Calculating the Mean Given a Frequency Distribution Table**

60 students were asked how many books they had read over the past 12 months. The results are listed in the frequency distribution table below. Calculate the mean number of books read by each student.
To determine the total number of books that were read by the students, each number of books must be multiplied by the number of students who read that particular number of books. Then all the products must be added to determine the total number of books read. This total number divided by 60 will tell you the mean number of books read by each student.

Thus, the mean number of books read by each student can be calculated as follows:
\[
\bar{x} = \frac{\sum x_1 f_1 + x_2 f_2 + x_3 f_3 + \ldots + x_n f_n}{f_1 + f_2 + f_3 + \ldots + f_n}
\]
\[
\bar{x} = \frac{\sum (0)(1) + (1)(6) + (2)(8) + (3)(10) + (4)(13) + (5)(8) + (6)(5) + (7)(6) + (8)(3)}{1 + 6 + 8 + 10 + 13 + 8 + 5 + 6 + 3}
\]
\[
\bar{x} = \frac{\sum 0 + 6 + 16 + 30 + 52 + 40 + 30 + 42 + 24}{60}
\]
\[
\bar{x} = \frac{240}{60}
\]
\[
\bar{x} = 4
\]

The mean number of books read by each student was 4 books.

**Determining the Mean Given a List**

Suppose the numbers of books read by each student in Example A were randomly listed as follows. Determine the mean of the numbers.

0 5 1 4 4 6 7 2 4 3 7 2 6 4 2
8 5 8 3 4 3 6 4 5 6 1 1 3 5 4
1 5 4 1 7 3 5 4 3 8 7 2 4 7 2
1 4 6 3 2 3 5 3 2 4 7 2 5 4 3

An alternative to entering all the numbers into a calculator would be to create a frequency distribution table like the one shown below:

<table>
<thead>
<tr>
<th>T A B L E 10.30:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Books</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>
Now that the data has been organized, the numbers of books read and the numbers of students who read the books are evident. The mean can now be calculated as it was in Example A:

\[
\overline{x} = \frac{\sum x_1 f_1 + x_2 f_3 + \ldots + x_n f_n}{f_1 + f_2 + f_3 + \ldots + f_n}
\]

\[
\overline{x} = \frac{\sum (0)(1) + (1)(6) + (2)(8) + (3)(10) + (4)(13) + (5)(8) + (6)(5) + (7)(6) + (8)(3)}{1 + 6 + 8 + 10 + 13 + 8 + 5 + 6 + 3}
\]

\[
\overline{x} = \frac{\sum 0 + 6 + 16 + 30 + 52 + 40 + 30 + 42 + 24}{60}
\]

\[
\overline{x} = \frac{240}{60}
\]

\[
\overline{x} = 4
\]

The mean number of books read by each student was 4 books.

**Using Technology**

Using technology, determine the mean of the following set of numbers:

24, 25, 25, 26, 26, 27, 27, 28, 28, 31, 32

Technology is a major tool that is available for you to use when doing mathematical calculations, and its use goes beyond entering numbers to perform simple arithmetic operations. For example, the TI-83 calculator can be used to determine the mean of a set of given data values. You will first learn to calculate the mean by simply entering the data values into a list and determining the mean. The second method that you will learn about utilizes the frequency table feature of the TI-83.

**Step 1:**

Stat → Enter → 2:SortA( → Enter → 3:SortD( → Enter → 4:ClrList → 5:SetUPEditor

**Step 2:**
Notice that the sum of the data values is 324 (∑x = 324).

Notice that the number of data values is 12 (n = 12).

Notice the mean of the data values is 27 (x̄ = 27).

Now we will use the same data values and use the TI-83 to create a frequency table.

Step 1:

Step 2:

Step 3:
Step 4:
Press

2ND

0

to obtain the CATALOG menu of the calculator. Scroll down to the sum function and enter $L_3 \rightarrow$

$$\text{sum}(L_3) = 324$$

You can repeat this step to determine the sum of $L_2 \rightarrow$

$$\text{sum}(L_2) = 12$$

Now the mean of the data can be calculated as follows:

$$\bar{x} = \frac{324}{12} = 27$$

Note that not all the data values and frequencies are visible in the screenshots, but rest assured that they were all entered into the calculator.

After entering the data into L1, the frequencies into L2, and pressing

2ND

MODE

, another way to solve this problem with the calculator would have been to press

2ND

STAT

, go to the MATH menu, choose option 3, and enter L1 and L2 separated by a comma so that you have mean(L1, L2). Then press

ENTER

to get the answer. This way, the calculator will do all the calculations for you.
Example

Example 1

The following data shows the heights in centimeters of a group of grade 10 students:

183 171 158 171 182 158 164 183
179 170 182 183 170 171 167 176
176 164 176 179 183 176 170 183
183 167 167 176 171 182 179 170

Organize the data in a frequency distribution table and calculate the mean height of the students.

The data can be organized into a frequency distribution table as shown below:

<table>
<thead>
<tr>
<th>Height of Students(cm)</th>
<th>Tally</th>
<th>Number of Students (Frequency)</th>
</tr>
</thead>
<tbody>
<tr>
<td>171</td>
<td></td>
<td></td>
</tr>
<tr>
<td>158</td>
<td></td>
<td></td>
</tr>
<tr>
<td>176</td>
<td></td>
<td></td>
</tr>
<tr>
<td>182</td>
<td></td>
<td></td>
</tr>
<tr>
<td>164</td>
<td></td>
<td></td>
</tr>
<tr>
<td>179</td>
<td></td>
<td></td>
</tr>
<tr>
<td>170</td>
<td></td>
<td></td>
</tr>
<tr>
<td>183</td>
<td></td>
<td></td>
</tr>
<tr>
<td>167</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now the frequency distribution table can be used to calculate the mean as follows:
The mean height of the students is approximately 174.1 cm.

**Review**

1. 45 students were asked how many e-mail messages they sent yesterday. The results are listed in the frequency distribution table below. Calculate the mean number of e-mail messages sent by each student.

<table>
<thead>
<tr>
<th>Number of E-Mail Messages</th>
<th>Number of Students (Frequency)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

2. 70 drivers were asked how many parking tickets they got last year. The results are listed in the frequency distribution table below. Calculate the mean number of parking tickets received by each driver.

<table>
<thead>
<tr>
<th>Number of Parking Tickets</th>
<th>Number of Drivers (Frequency)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
</tr>
</tbody>
</table>

3. The following data shows the numbers of siblings of a group of grade 10 students. Organize the data in a
frequency distribution table and calculate the mean number of siblings of the students.

\[
\begin{align*}
2 & 1 3 0 2 3 0 4 \\
0 & 1 3 2 4 0 0 1 \\
1 & 1 4 3 2 1 0 1 \\
5 & 2 1 2 1 3 4 6 \\
\end{align*}
\]

4. The following data shows the numbers of touchdowns scored by each of the teams in the National Football League last week. Organize the data in a frequency distribution table and calculate the mean number of touchdowns scored by the teams.

\[
\begin{align*}
3 & 1 0 5 7 4 3 2 \\
4 & 3 3 4 3 2 5 4 \\
2 & 2 2 4 3 4 4 1 \\
2 & 8 3 2 0 1 5 4 \\
\end{align*}
\]

A set of data values was entered into L1 on a TI calculator, and 1-Var Stats returned the results shown below:

\[
\begin{align*}
\bar{x} &= 9 \\
\sum x &= 108 \\
\sum x^2 &= 1174 \\
\mu &= 4.285281363 \\
\sigma &= 4.102844542 \\
n &= 12 \\
\end{align*}
\]

5. What is the sum of the data values?
6. How many data values are there?
7. What is the mean of the data values?

8. The following frequency distribution table was entered into a TI calculator. All the data values and frequencies are visible in the screenshot.

What is the mean of the data?

9. The following frequency distribution table was entered into a TI calculator. All the data values and frequencies are visible in the screenshot.
What is the mean of the data?

10. The following frequency distribution table was entered into a TI calculator. All the data values and frequencies are visible in the screenshot.

Review (Answers)

To view the Review answers, open this PDF file and look for section 5.2.
10.16 Grouped Data to Find the Mean

Here you’ll learn how to organize and use a frequency distribution table to calculate the mean or average of a set of data that is grouped in intervals.

**Grouped Data to Find the Mean**

All of the values for the means that you have calculated so far have been for ungrouped, or listed, data. A mean can also be determined for grouped data, or data that is placed in intervals. Unlike listed data, the individual values for grouped data are not available, and you are not able to calculate their sum. To calculate the mean of grouped data, the first step is to determine the midpoint of each interval, or class. These midpoints must then be multiplied by the frequencies of the corresponding classes. The sum of the products divided by the total number of values will be the value of the mean.

In other words, the mean for a population can be found by dividing $\sum m \cdot f$ by $N$, where $m$ is the midpoint of the class and $f$ is the frequency. As a result, the formula $\mu = \frac{\sum m \cdot f}{N}$ can be written to summarize the steps used to determine the value of the mean for a set of grouped data. If the set of data represented a sample instead of a population, the process would remain the same, and the formula would be written as $\bar{x} = \frac{\sum m \cdot f}{n}$.

The following examples will show how the mean value for grouped data can be calculated.

**Calculating the Mean**

1. In Tim’s school, there are 25 teachers. Each teacher travels to school every morning in his or her own car. The distribution of the driving times (in minutes) from home to school for the teachers is shown in the table below:

<table>
<thead>
<tr>
<th>Driving Times (minutes)</th>
<th>Number of Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to less than 10</td>
<td>3</td>
</tr>
<tr>
<td>10 to less than 20</td>
<td>10</td>
</tr>
<tr>
<td>20 to less than 30</td>
<td>6</td>
</tr>
<tr>
<td>30 to less than 40</td>
<td>4</td>
</tr>
<tr>
<td>40 to less than 50</td>
<td>2</td>
</tr>
</tbody>
</table>
The driving times are given for all 25 teachers, so the data is for a population. Calculate the mean of the driving times.

**Step 1:** Determine the midpoint for each interval.
- For 0 to less than 10, the midpoint is 5.
- For 10 to less than 20, the midpoint is 15.
- For 20 to less than 30, the midpoint is 25.
- For 30 to less than 40, the midpoint is 35.
- For 40 to less than 50, the midpoint is 45.

**Step 2:** Multiply each midpoint by the frequency for the class.
- For 0 to less than 10, \((5)(3) = 15\)
- For 10 to less than 20, \((15)(10) = 150\)
- For 20 to less than 30, \((25)(6) = 150\)
- For 30 to less than 40, \((35)(4) = 140\)
- For 40 to less than 50, \((45)(2) = 90\)

**Step 3:** Add the results from Step 2 and divide the sum by 25.

\[
15 + 150 + 150 + 140 + 90 = 545 \\
\mu = \frac{545}{25} = 21.8
\]

Each teacher spends a mean time of 21.8 minutes driving from home to school each morning.

To better represent the problem and its solution, a table can be drawn as follows:

<table>
<thead>
<tr>
<th>Driving Times (minutes)</th>
<th>Number of Teachers (f)</th>
<th>Midpoint Of Class (m)</th>
<th>Product (mf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to less than 10</td>
<td>3</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>10 to less than 20</td>
<td>10</td>
<td>15</td>
<td>150</td>
</tr>
<tr>
<td>20 to less than 30</td>
<td>6</td>
<td>25</td>
<td>150</td>
</tr>
<tr>
<td>30 to less than 40</td>
<td>4</td>
<td>35</td>
<td>140</td>
</tr>
<tr>
<td>40 to less than 50</td>
<td>2</td>
<td>45</td>
<td>90</td>
</tr>
</tbody>
</table>

For the population, \(N = 25\) and \(\Sigma mf = 545\), so using the formula \(\mu = \frac{\Sigma mf}{N}\), the mean would again be \(\mu = \frac{545}{25} = 21.8\).

2. In the previous example, suppose the distribution of driving times were broken down into smaller intervals as shown:

<table>
<thead>
<tr>
<th>Driving Times (minutes)</th>
<th>Number of Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to less than 5</td>
<td>2</td>
</tr>
<tr>
<td>5 to less than 10</td>
<td>1</td>
</tr>
<tr>
<td>10 to less than 15</td>
<td>4</td>
</tr>
<tr>
<td>15 to less than 20</td>
<td>6</td>
</tr>
<tr>
<td>20 to less than 25</td>
<td>3</td>
</tr>
<tr>
<td>25 to less than 30</td>
<td>3</td>
</tr>
</tbody>
</table>
Calculate the mean of the driving times.

First create the table below:

<table>
<thead>
<tr>
<th>Driving Times (minutes)</th>
<th>Number of Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 to less than 35</td>
<td>1</td>
</tr>
<tr>
<td>35 to less than 40</td>
<td>3</td>
</tr>
<tr>
<td>40 to less than 45</td>
<td>1</td>
</tr>
<tr>
<td>45 to less than 50</td>
<td>1</td>
</tr>
</tbody>
</table>

Now the mean can be calculated as shown:

\[
\mu = \frac{\sum mf}{N}
\]

\[
\mu = \frac{5.0 + 7.5 + 50.0 + 105.0 + 67.5 + 82.5 + 32.5 + 112.5 + 42.5 + 47.5}{25}
\]

\[
\mu = \frac{552.5}{25}
\]

\[
\mu = 22.1
\]

This time, the mean time spent by each teacher driving from home to school is 22.1 minutes. Thus, the mean for grouped data can change based on the size of the intervals.

3. The ages of 100 singers of a 360-member choir are shown in the table below:

<table>
<thead>
<tr>
<th>Ages of Members (years)</th>
<th>Number of Members</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 to less than 25</td>
<td>12</td>
</tr>
<tr>
<td>25 to less than 30</td>
<td>14</td>
</tr>
<tr>
<td>30 to less than 35</td>
<td>10</td>
</tr>
<tr>
<td>35 to less than 40</td>
<td>8</td>
</tr>
<tr>
<td>40 to less than 45</td>
<td>20</td>
</tr>
<tr>
<td>45 to less than 50</td>
<td>6</td>
</tr>
<tr>
<td>50 to less than 55</td>
<td>5</td>
</tr>
<tr>
<td>55 to less than 60</td>
<td>4</td>
</tr>
</tbody>
</table>
Calculate the mean of the ages.

First create the table below:

<table>
<thead>
<tr>
<th>Ages of Members (years)</th>
<th>Number of Members</th>
<th>Midpoint Of Class $m$</th>
<th>Product $mf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 to less than 25</td>
<td>12</td>
<td>22.5</td>
<td>270.0</td>
</tr>
<tr>
<td>25 to less than 30</td>
<td>14</td>
<td>27.5</td>
<td>385.0</td>
</tr>
<tr>
<td>30 to less than 35</td>
<td>10</td>
<td>32.5</td>
<td>325.0</td>
</tr>
<tr>
<td>35 to less than 40</td>
<td>8</td>
<td>37.5</td>
<td>300.0</td>
</tr>
<tr>
<td>40 to less than 45</td>
<td>20</td>
<td>42.5</td>
<td>850.0</td>
</tr>
<tr>
<td>45 to less than 50</td>
<td>6</td>
<td>47.5</td>
<td>285.0</td>
</tr>
<tr>
<td>50 to less than 55</td>
<td>5</td>
<td>52.5</td>
<td>262.5</td>
</tr>
<tr>
<td>55 to less than 60</td>
<td>4</td>
<td>57.5</td>
<td>230.0</td>
</tr>
<tr>
<td>60 to less than 65</td>
<td>11</td>
<td>62.5</td>
<td>687.5</td>
</tr>
<tr>
<td>65 to less than 70</td>
<td>10</td>
<td>67.5</td>
<td>675.0</td>
</tr>
</tbody>
</table>

Since the ages represent a sample, the mean can be calculated as shown:

$$\bar{x} = \frac{\sum mf}{n}$$

$$\bar{x} = \frac{270.0 + 385.0 + 325.0 + 300.0 + 850.0 + 285.0 + 262.5 + 230.0 + 687.5 + 675.0}{100}$$

$$\bar{x} = \frac{4,270.0}{100}$$

$$\bar{x} = 42.7$$

The mean age of the 100 members of the choir is 42.7 years.

Points to Consider

- Is the mean only used as a measure of central tendency, or is it applied to other representations of data?
- If the mean is applied to other representations of data, can its value be calculated or estimated from this representation?
- What other measures of central tendency can be used as a statistical summary when the mean is not the best measure to use?
Examples

The following table shows the frequency distribution of the number of hours spent per week texting messages on a cell phone by 60 grade 10 students at a local high school.

<table>
<thead>
<tr>
<th>Time Per Week (Hours)</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to less than 5</td>
<td>8</td>
</tr>
<tr>
<td>5 to less than 10</td>
<td>11</td>
</tr>
<tr>
<td>10 to less than 15</td>
<td>15</td>
</tr>
<tr>
<td>15 to less than 20</td>
<td>12</td>
</tr>
<tr>
<td>20 to less than 25</td>
<td>9</td>
</tr>
<tr>
<td>25 to less than 30</td>
<td>5</td>
</tr>
</tbody>
</table>

**Table 10.40:**
Example 1

Calculate the mean number of hours per week spent by each student texting messages on a cell phone. Hint: A table may be useful.

First create the table below.

<table>
<thead>
<tr>
<th>Time Per Week (Hours)</th>
<th>Number of Students $f$</th>
<th>Midpoint of Class $m$</th>
<th>Product $mf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to less than 5</td>
<td>8</td>
<td>2.5</td>
<td>20.0</td>
</tr>
<tr>
<td>5 to less than 10</td>
<td>11</td>
<td>7.5</td>
<td>82.5</td>
</tr>
<tr>
<td>10 to less than 15</td>
<td>15</td>
<td>12.5</td>
<td>187.5</td>
</tr>
<tr>
<td>15 to less than 20</td>
<td>12</td>
<td>17.5</td>
<td>210.0</td>
</tr>
<tr>
<td>20 to less than 25</td>
<td>9</td>
<td>22.5</td>
<td>202.5</td>
</tr>
<tr>
<td>25 to less than 30</td>
<td>5</td>
<td>27.5</td>
<td>137.5</td>
</tr>
</tbody>
</table>

Now that you have created several distribution tables for grouped data, it’s time to point out that the first column of the table can be represented in another way. As an alternative to writing the interval, or class, in words, the words can be expressed as [# - #), where the front square bracket closes the class, so the first number is included in the designated interval, but the open bracket at the end does not close the class, so the last number is not included in the designated interval. Keeping this in mind, the table above can be presented as follows:

<table>
<thead>
<tr>
<th>Time Per Week (Hours)</th>
<th>Number of Students $f$</th>
<th>Midpoint of Class $m$</th>
<th>Product $mf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 – 5)</td>
<td>8</td>
<td>2.5</td>
<td>20.0</td>
</tr>
<tr>
<td>(5 – 10)</td>
<td>11</td>
<td>7.5</td>
<td>82.5</td>
</tr>
<tr>
<td>(10 – 15)</td>
<td>15</td>
<td>12.5</td>
<td>187.5</td>
</tr>
<tr>
<td>(15 – 20)</td>
<td>12</td>
<td>17.5</td>
<td>210.0</td>
</tr>
<tr>
<td>(20 – 25)</td>
<td>9</td>
<td>22.5</td>
<td>202.5</td>
</tr>
<tr>
<td>(25 – 30)</td>
<td>5</td>
<td>27.5</td>
<td>137.5</td>
</tr>
</tbody>
</table>

Now the mean can be calculated as shown:
The mean time spent per week by each student texting messages on a cell phone is 14 hours.

Review

1. Match the words in the left column with the correct symbol from the right column.

1. Sample mean  
   2. The sum of  
   3. Population mean  
   4. Number of data for a sample  
   5. Product of the midpoint and the frequency  
   6. Number of data for a population

A. $mf$  
B. $N$  
C. $\bar{x}$  
D. $\mu$  
E. $n$  
F. $\Sigma$

The following table of grouped data represents the weights (in pounds) of all 100 babies born at a local hospital last year.

**Table 10.43:**

<table>
<thead>
<tr>
<th>Weight (pounds)</th>
<th>Number of Babies</th>
</tr>
</thead>
<tbody>
<tr>
<td>[3 – 5)</td>
<td>8</td>
</tr>
<tr>
<td>[5 – 7)</td>
<td>25</td>
</tr>
<tr>
<td>[7 – 9)</td>
<td>45</td>
</tr>
<tr>
<td>[9 – 11)</td>
<td>18</td>
</tr>
<tr>
<td>[11 – 13)</td>
<td>4</td>
</tr>
</tbody>
</table>

2. What is the summation of the values of $mf$ for each class?
3. What is the value of $N$?
4. Calculate the mean weight for a baby.

The following table of grouped data represents the ages (in years) of 50 of the 500 teachers in a school district.
Table 10.44:

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Number of Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>[25 – 35)</td>
<td>7</td>
</tr>
<tr>
<td>[35 – 45)</td>
<td>8</td>
</tr>
<tr>
<td>[45 – 55)</td>
<td>16</td>
</tr>
<tr>
<td>[55 – 65)</td>
<td>10</td>
</tr>
<tr>
<td>[65 – 75)</td>
<td>9</td>
</tr>
</tbody>
</table>

5. What is the summation of the values of $mf$ for each class?
6. What is the value of $n$?
7. Calculate the mean age for a teacher.

The following table of grouped data represents the numbers of miles on all 25 cars in a used car lot.

Table 10.45:

<table>
<thead>
<tr>
<th>Mileage</th>
<th>Number of Cars</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0 – 40,000)</td>
<td>3</td>
</tr>
<tr>
<td>[40,000 – 80,000)</td>
<td>6</td>
</tr>
<tr>
<td>[80,000 – 120,000)</td>
<td>10</td>
</tr>
<tr>
<td>[120,000 – 160,000)</td>
<td>4</td>
</tr>
<tr>
<td>[160,000 – 200,000)</td>
<td>2</td>
</tr>
</tbody>
</table>

8. What is the summation of the values of $mf$ for each class?
9. What is the value of $N$?
10. Calculate the mean number of miles on a car.

Review (Answers)

To view the Review answers, open this PDF file and look for section 5.3.
Here you’ll learn how to use a Texas Instruments calculator to find the median of a data set.

**Median of Large Sets of Data**

Often, the number of data values is quite large, and the task of organizing the data can take a great deal of time. To help organize data, the TI-83 calculator can be used. The following examples will show you how to use the calculator to organize data and find the median for the data values.

**Finding the Median Using a Calculator**

The local police department spent the holiday weekend ticketing drivers who were speeding. 50 locations within the state were targeted as being ideal spots for drivers to exceed the posted speed limit. The number of tickets issued during the weekend in each of the locations is shown in the following table. What is the median number of speeding tickets issued? Use your TI calculator to find the answer.

```
32  12  15  8  16  42  9  18  11  10
24  18  6  17  21  41  3  5  35  27
13  26  16  28  31  3  7  37  10  19
23  33  7  25  36  40  15  21  38  46
17  37  9  2  33  41  23  29  19  40
```

Using the TI-83 calculator:
Step 1:

STAT → 1:Edit... 2:SortA( 3:SortD( 4:ClrList 5:SetUpEditor

Step 2:

STAT → 2nd 1 → SortA(L1

The numbers that you entered into L1 are now sorted from smallest to largest.

Step 3:
You can now scroll down the list to reveal the ordered numbers.

There are 50 data values in the table. The median will be the mean of the number before the \( \frac{n+1}{2} \) position and the number after the \( \frac{n+1}{2} \) position: \( \frac{50+1}{2} = \frac{51}{2} = 25.5 \). The number before the 25.5 position is 19, and the number after the 25.5 position is 21. This means that the median is \( \frac{19+21}{2} = \frac{40}{2} = 20 \).


\[
\begin{array}{cccccccccccc}
2 & 3 & 3 & 5 & 6 & 7 & 7 & 8 & 9 & 9 \\
10 & 10 & 11 & 12 & 13 & 15 & 15 & 16 & 16 & 17 \\
17 & 18 & 18 & 19 & 21 & 21 & 23 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 31 & 32 & 33 & 33 & 35 \\
36 & 37 & 37 & 38 & 40 & 40 & 41 & 41 & 42 & 46 \\
\end{array}
\]

**Calculating the Median Without Sorting Data**

Calculate the median in the previous example without sorting the data.

The following are 2 ways that you can use the TI-83 to determine the median of the values without sorting the data.

**Method One:**

All of the data values have been entered into L1. The median of the data values can now be determined by using the TI-83 as follows:

```
STAT → EDIT → 1:EditL1
L1 → 2nd 1

Med = 20 indicates that the median of the data values in L1 is 20.
```

**Method Two:**

Above the 0 key is the word CATALOG, and this function acts like the yellow pages of a telephone book. When you press

```
2ND
0
```

to access the CATALOG menu, an alphabetical list of terms appears. You can either scroll down to the word median (this will take a long time) or press the blue ÷ to access all terms beginning with the letter ‘m’.

```
2nd 0 CATALOG median(L1) Enter 2nd 1
```

Scroll down to median(L1) Enter 2nd 1 Enter 20
Again, the median of the values in L1 is 20.
Whichever method you use, the result will be the same. Using technology will save you time when you are determining the median of a set of data values.

**Example C**

The following data values represent the numbers of chocolate bars sold by students during a recent fundraising campaign. What was the median number of chocolate bars sold? Use your TI calculator to find the answer.

**Table 10.46:**

<table>
<thead>
<tr>
<th>Number of Bars</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
</tr>
<tr>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>16</td>
<td>15</td>
</tr>
</tbody>
</table>

When a set of data values is given in the form of a **frequency table**, technology is often used to determine the median. Using the TI-83 calculator:

![TI-83 Calculator Screen](https://www.ck12.org/flx/render/embeddedobject/137049)

The median number of chocolate bars sold was 14.5.

Notice that in this example, each of the individual data values was entered into LI to calculate the median.
Example

Example 1

Calculate the median for the data in the last example by entering the frequency table and not each of the individual data values into your calculator. The data is shown again below:

<table>
<thead>
<tr>
<th>Number of Bars</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
</tr>
<tr>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>16</td>
<td>15</td>
</tr>
</tbody>
</table>

First, the frequency table can be entered into L1 and L2 as follows:

Next, press

2ND

MODE

2ND

STAT

and go to the MATH menu:
Choose median by pressing

4

or by using the down arrow and pressing

ENTER

, and then enter L1 and L2 separated by a comma and close the parentheses so that you have median(L1, L2):

\[
\text{median}(L1, L2)
\]

Finally, press

ENTER

to calculate the answer:

\[
\text{median}(L1, L2) = 14.5
\]

**Review**

1. A minor hockey league has 50 active players who range in age from 10 years to 15 years. The following table shows the ages of the players:
TABLE 10.48:

<table>
<thead>
<tr>
<th>Age of Players (yrs)</th>
<th>Number of Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>12</td>
<td>7</td>
</tr>
<tr>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
</tr>
</tbody>
</table>

Use your TI calculator to determine the median age of the players.

2. A die was thrown 14 times, and the results of each throw are shown:

![Die roll results]

3. At a local golf club, 100 players competed in a 1-day tournament. The fifth hole of the course is a par 6. The scores of each player on this hole were recorded, and the results are shown below:

![Golf course]

<table>
<thead>
<tr>
<th>Score</th>
<th>Number of Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td>29</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

Use your TI calculator to find the median score for the players.

4. A math teacher at a local high school begins every class with a warm-up quiz based on the work presented the previous day. Each test consists of 6 questions valued at 1 point for each correct answer. The results of Monday’s quiz for the 40 students in the class are shown below:

![Quiz results]

![Hand with pen and paper]

Use your TI calculator to determine the median score for the students.
What is the median score for Monday’s quiz? Use your TI calculator to find the answer.

5. In Canada, with the loonie and the toonie, you could have a lot of coins in your pocket. A number of high school students were asked how many coins they had in their pocket, and the results are shown in the following table:

<table>
<thead>
<tr>
<th>Number of Coins</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Students</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>6</td>
<td>4</td>
<td>9</td>
<td>7</td>
</tr>
</tbody>
</table>

Use your TI calculator to find the median number of coins that a student had in his or her pocket.

6. A group of 12 students participated in a local dirt bike race that required them to cover a 1-mile course in the fastest time possible. The times, in minutes, of the 12 participants are shown below:

3.5 min 4.2 min 3.1 min 5.3 min 6.2 min 4.6 min 5.1 min 6.7 min 5.4 min 4.4 min 3.9 min 5.0 min

What is the median time of the participants in the race? Use your TI calculator to find the answer.

For the data in each of the following frequency tables, find the median. All of the data is shown in each of the screenshots.
Review (Answers)

To view the Review answers, open this PDF file and look for section 5.5.

Summarize and describe distributions.

6.SP.4 Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

6.SP.5 Summarize numerical data sets in relation to their context, such as by:

a. Reporting the number of observations.

b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.

c. Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.